A. Verruijt (Technische Hogeschool Delft)

ANALYSIS OF INTERFACE PROBLEMS BY THE FINITE ELEMENT METHOD.

1. INTRODUCTION

In recent years the finite element method has been found to be a powerful tool for the solution of groundwater flow problems , see for instance Zienkiewicz and Cheung (1). In this method an approximate solution is determined in the form of values of the groundwater head in a great (but finite) number of points. These values are calculated by means of a computer program which in general solves a system of linear algebraic equations. As the construction of the system of equations is also performed within the computer program, the finite element method enables the solution of complicated problems, such as problems for anisotropic, non-homogeneous aquifers of arbitrary shape. The occurrence of a free surface or a salt water interface can be taken into account in a relatively simple way.

Advantages of the finite element method are

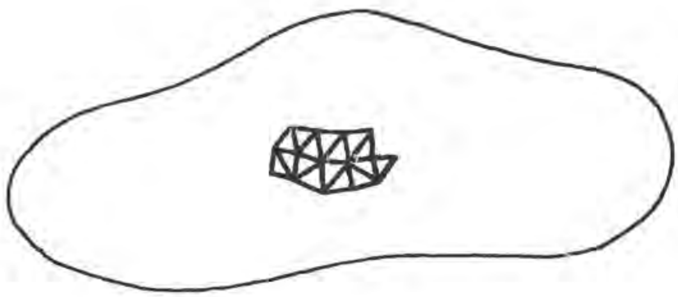
1. Solution possible for soil bodies of arbitrary geometrical shape.
2. Boundary conditions may be complicated (infiltration, leakage, free surface, interface).
3. Non-homogeneity of the soil is easily taken into account.
4. Anisotropy is only a very small complication.
5. Generalization to non-steady flow is relatively simple.

Disadvantages of the method are

1. The theoretical foundation is somewhat abstract (variational principle).
2. The computer program is fairly complicated (this is the price that one has to pay for generality and flexibility).
3. For problems involving many points and complicated boundary conditions the requirements on the computer's memory and time may surpass the limits of economic feasibility.

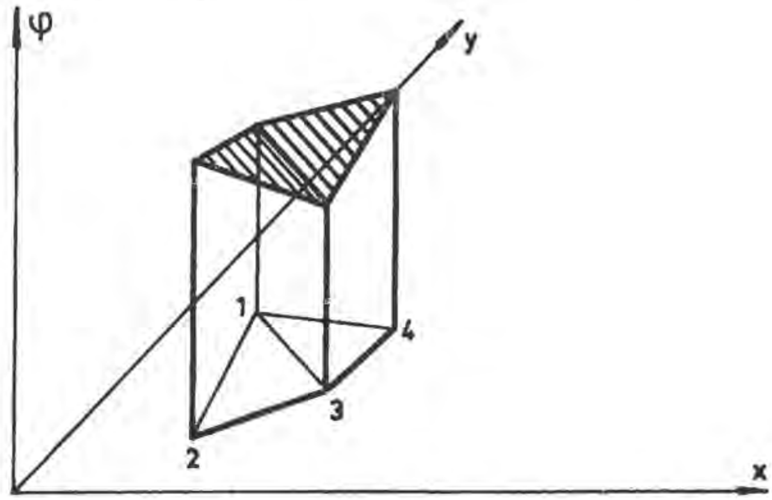
2. DESCRIPTION OF THE FINITE ELEMENT METHOD

For a detailed derivation of the mathematical equations involved in the finite element method the reader is referred to the literature, for instance Zienkiewicz and Cheung (1), Verruijt (2,3). Here only a general description is presented. Restriction is made to flow in a plane, see fig. 1.



Figuur 1

The plane is covered by a network of triangles, and it is assumed that within each triangle the variation of the head is linear, see fig. 2.



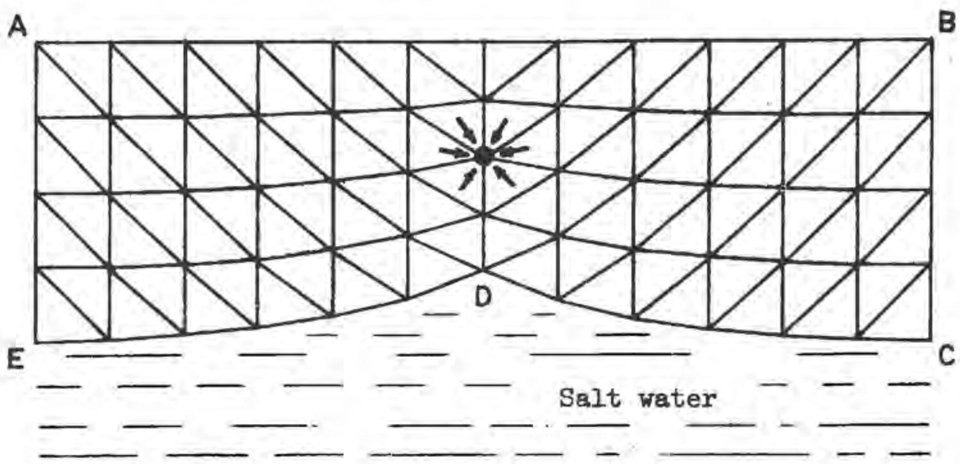
Figuur 2

This means that the function $\varphi = \varphi(x,y)$ is approximated by a piecewise differentiable function, represented by a surface of facets. The values of the groundwater head φ in each node are considered as unknowns. Thus the continuous variable $\varphi(x, y)$ is replaced by a great number (say 100 or 200, or 1000) of unknowns $\varphi_k \ (k = 1, 2, \ldots, n)$. Instead of the differential equation governing the function $\varphi(x, y)$ one now uses a system of linear algebraic equations that is supposed to be equivalent to the differential equation. These equations are derived by requiring that the approximate solution on the average approximates the true solution as good as possible. The system of equations can be set up by a computer program that uses the following data as input:

1. The coordinates of each node,
2. The structure of each element,
3. The transmissibility in each element,
4. An initial estimation of the head,
5. For non-steady flow: the storage coefficient, and the values of time-steps.

The computer then calculates the coefficients of the algebraic equations, and determines the solution, for instance by an iterative method similar to the usual relaxation process.

The occurrence of a free surface or an interface leads to a complication that can be overcome fairly easily, although a considerable increase in computer time has to be accepted then. The principle of the procedure can best be explained by means of a simple example, see fig. 3.



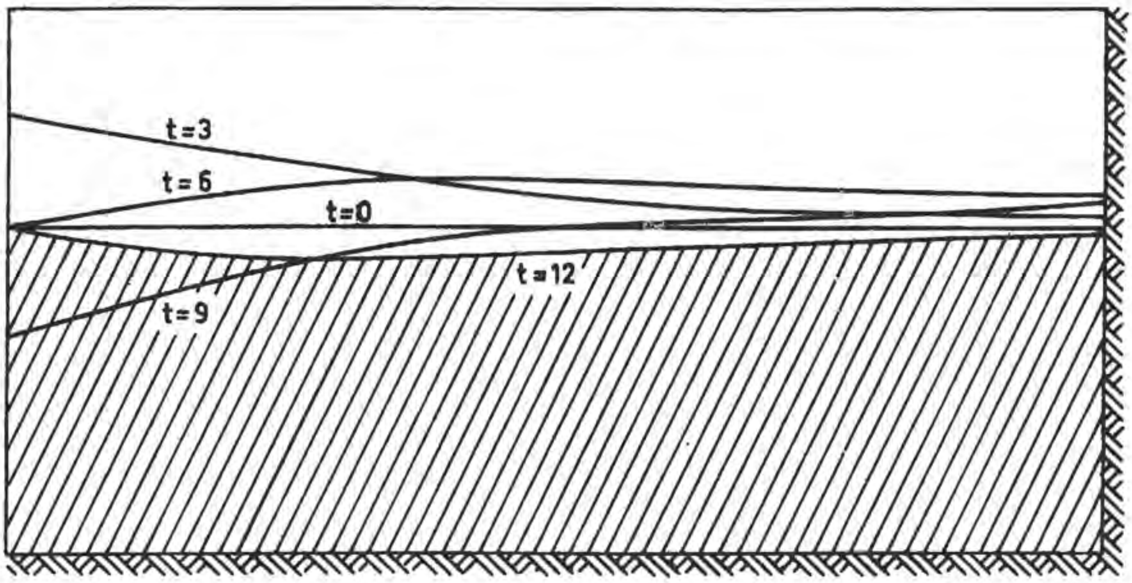
Figuur 3

The flow region at a certain moment (say $t = t_0$) is subdivided into triangles. It is assumed that at all times the salt water head in the salt water is constant (a more refined technique would involve a network of triangles in the salt water also; this is possible, but has not yet been executed). Then, since at both sides of the interface the fluid pressure must be the same, the fresh water head along CDE is known. Suppose that along AB the head is constant, and let AE and BC be sufficiently far away to assume that along AE and BC the head is constant. Then along the entire boundary the head is given. The finite element procedure then enables to calculate the values of the head in interior points, and the velocities in each point. The velocities in the points on the interface multiplied by Δt yield the displacements of these points during the interval Δt. This then enables the calculation of the coordinates of the nodal points along CDE at the end of the time interval. On the basis of these new coordinates the computer can next calculate the new values of the coefficients of the linear equations, solve the system, etc.

3. EXAMPLES

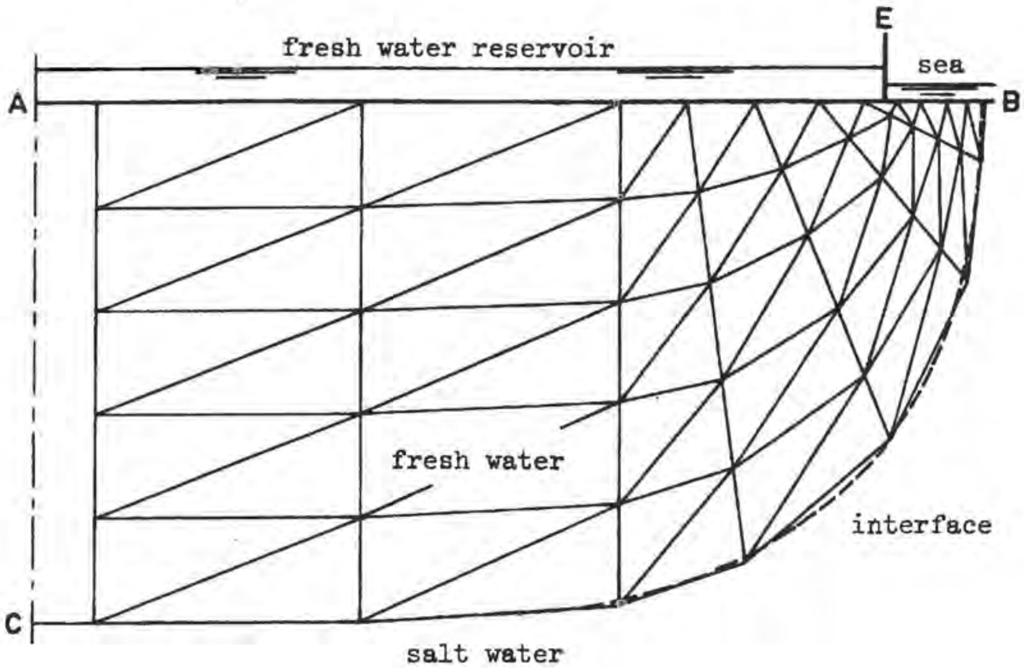
Some examples are given below, in the form of graphical representations of solutions.

The first example, taken from Verruijt (2), refers to non-steady flow with a free surface in a rectangular dam. The response of the free surface to a sinusoidal variation of the head on the left side is shown in fig. 4. The results have been compared with other approximate solutions, and with results of tests in a Hele-Shaw model. It appears that a sufficient accuracy can be obtained provided that the time steps are taken small enough.



Figuur 4

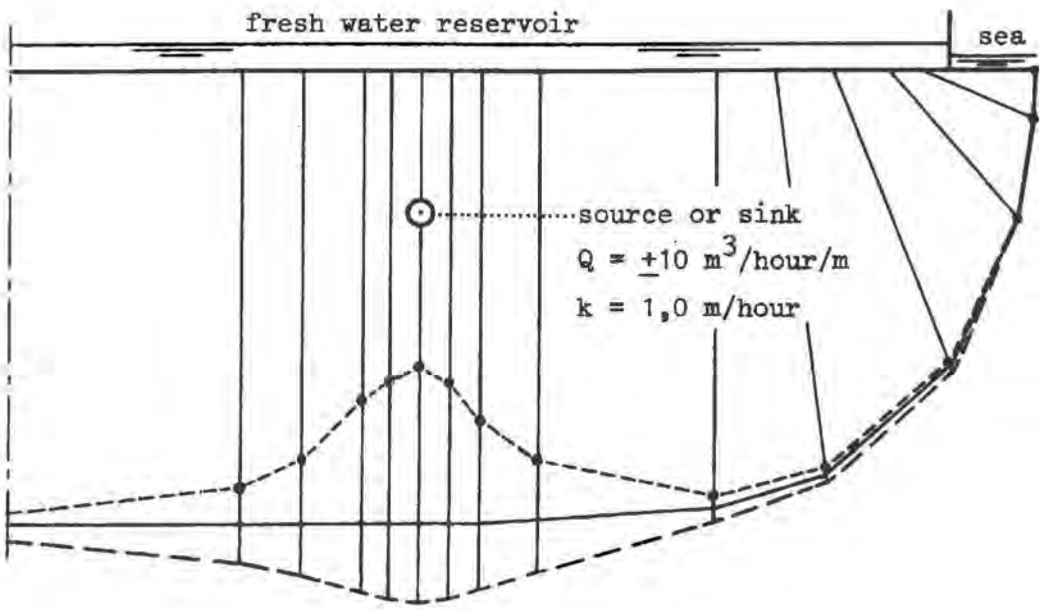
The second example refers to steady flow with an interface between fresh and salt water, see fig. 5. The fresh water



Figuur 5

is flowing from a fresh water reservoir (or a high land area) towards the sea. The salt water is supposed to be stationary. The theoretical solution for this problem was obtained by Strack (4), with the aid of complex variables. In this theoretical solution the upper boundary was schematized to a straight potential line. At great distances from the sea the depth of the interface is, of course, determined by the Ghyben-Herzberg formula. The numerical solution by means of the finite element method was obtained by Kono (5). The final mesh of triangles is shown in fig. 5. The dashed line represents the theoretical position of the interface as calculated by means of the exact complex variable method.

The advantage of a finite element solution as compared to an analytical solution is that it is easier to generalize the finite element technique. It is a relatively simple matter to include non-homogeneities in the soil permeability, etcetera. As an example some results for the same situation as in fig. 5, with a sink or a source in the field, are represented in fig. 6.



Figuur 6

The dashed line is the position of the interface in case of a source, and the dotted line is the interface in case of a sink. The program to calculate the interface was written by Kono (5). Although this problem admits a theoretical solution no comparisons were made as yet. It might be mentioned, in conclusion, that in general the location of the nodes changes during the iterative determination of the position of the interface. One may specify, for instance, that on certain vertical lines 6 nodes are located between the fixed upper boundary and the lower boundary, the interface, the position of which changes during the calculation procedure. Thus at every stage of the calculations the interface consists of the lowest nodal points.

Literature

(1) Zienkiewics, O.C., and Cheung, Y.K., The Finite Element Method in Structural and Continuum Mechanics, McGraw-Hill, London, 1967.

(2) Verruijt, A., Theory of Groundwater Flow, Macmillan, London, 1970.

(3) Verruijt, A., A Finite Element Approach to Plane Groundwater Flow Problems with Storage, Infiltration and Leakage, University Soil Mechanics Laboratory Report, Delft, 1972.

(4) Strack, O.D.L., Some Cases of Interface Flow towards Drains, University Soil Mechanics Laboratory Report, Delft, 1971.

(5) Kono, I., Analysis of Interface Problems in Groundwater Flow by Finite Element Method, University Soil Mechanics Laboratory Report, Delft, 1972.