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A FINITE DIFFERENCE MODEL FOR THE COMPUTER SIMULATION OF SALTWATER INTRUSION IN COASTAL AQUIFERS

SUMMARY

In this paper we present and discuss a finite difference model for the computer simulation of transient phenomena of salt water intrusion in coastal aquifers. The finite difference equations are derived from the physical principles of conservation of momentum, mass and volume. The resulting algorithm is fast, accurate and applies to problems with very general boundary configurations. It is shown that problems with numerical singularities, such as those which arise in the simulation of injection or extraction wells, can be treated accurately by using a nonuniform mesh. The numerical results given illustrate some of the uses and the flexibility of this computing technique.

1. GOVERNING EQUATIONS

Consider a section of a homogenous isotropic aquifer that is cut vertically downward and perpendicular to the coast. The fluid motion is assumed to be two dimensional. The aquifer has porosity $\varepsilon(x,y)$ and permeability $K(x,y)$ and we assume that there is also a source term S as an extraction or injection well. We denote by ρ_S the density of the source fluid, which is assumed to be

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known. Under these conditions seawater encroachment can be described by the conservation equations of mass, linear momentum and volume.

The equation for the conservation of mass is given by (see, e.g., [1]):

$$\varepsilon \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = \tau \left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right) + (\rho_S - \rho) S \tag{1}$$

where $\rho(x,y,t)$ is the density of the fluid, t the time, $u(x,y,t)$, $v(x,y,t)$ are the components of the seepage velocity vector in the x-and y-directions, respectively, and $\tau(x,y)$ is the mass diffusivity.

In order to solve equation (1) the knowledge of the seepage velocity field $u(x,y,t)$, $w(x,y,t)$ is required.

The equations for linear momentum, and the equation for the conservation of linear momentum, and the equation for the conservation of volume are given, respectively, by (see, e.g., [2]):

$$\frac{\rho}{\varepsilon} \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x} - \frac{\mu}{k} u$$

$$\frac{\rho}{\varepsilon} \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial x} - \frac{\mu}{k} v - \rho g \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{dv}{\partial y} = S \ ,$$

where p is the pressure, μ is the viscosity of the fluid, and g is the gravity acceleration vector.

The system of equations (1), (2) is a system of four equations with four unknowns ρ, u, v, p.

If there exists a free surface, representing the surface of separation between liquid and gas, which can be represented as a single valued function $y = h(x,t)$, the change in surface elevation is determined by the following kinematic condition

$$\varepsilon_s \frac{\partial h}{\partial t} + u_s \frac{\partial h}{\partial x} = v_s \tag{3}$$

where $\varepsilon_s(x,y)$ is the porosity of the medium at free surface, and $u_s(x,y)$ and $v_s(x,y)$ are the seepage velocity components at the fluid surface.

The following initial conditions are associated with (1), (2) and (3).

$$\begin{aligned} \rho(x,y,t_o) &= \rho_o(x,y) \\ u(x,y,t_o) &= u_o(x,y) \\ v(x,y,t_o) &= v_o(x,y) \\ h(x,t_o) &= h_o(x) \end{aligned} \tag{4}$$

The boundary conditions for (1) are given as follows. Either the density ρ or its normal derivative is given at the boundary, unless the mass diffusivity τ is zero, in which case the density ρ or its normal derivative has to be given at the inflow boundary only. The boundary conditions for (2) require either the pressure p_s or the normal velocity $q_\perp$ given as function of time. The free surface equation (3) needs one boundary condition at the point of intersection with the fixed boundary only if the fixed boundary is an inflow boundary at such point. In this case the surface height must be specified at this point as a function of time.

2. FINITE DIFFERENCE DISCRETIZATION

The approximation technique we have used to solve equation (1) considers three possible physical cases. In the first case, the convective terms are considered to be so large that equation (1) behaves like a hyperbolic equation.

In the second case, the transport due to the convection is considered to be comparable to that due to dispersion, under some suitable assumption on τ, assumed to be strictly positive, so the equation (1) behaves like a parabolic equation.

In the third case, the convective transport and dispersive transport are considered simultaneously without any assumption about their reciprocal order of magnitude.

The approximation of the system (2), that we have used, is based on the M.A.C. method [6] and on several techniques, developed by the present writers, which allow the treatment of very general boundary configuration and the use of different space steps (see, e.g., [1], [2], [3]). More precisely, if $D \subset \mathbb{R}^2$ is the domain for the system (1), (2), the finite difference mesh used to discretize equations (1) - (2), consists of rectangular cells of width Δx and height Δy which cover the domain D. The field variables u, v, p and ρ are defined as follows: u-velocity at the center of each vertical side of a cell, v-velocity at the center of each horizontal side of a cell and pressure p and density ρ at each cell center. Each cell will be numbered at its center by indices i and j, where i is the cell's column number in x direction and j is the cell's row number in y direction. We denote by Δt the discrete time step.

So we consider in each cell (i, j) for the equation (1) three different finite difference schemes corresponding to the types of behavior described above:

$$\varepsilon_{i,j} \quad \rho^{n+1}_{i,j} = \frac{1}{4}\left(\rho^n_{i+1,j} + \rho^n_{i-1,j} + \rho^n_{i,j+1} + \rho^n_{i,j-1}\right) - \tag{5}$$
$$- \Delta t \left(u^n_{i,j} \frac{1}{2\Delta x} \left(\rho^n_{i+1,j} - \rho^n_{i-1,j}\right) + v^n_{i,j} \frac{1}{2\Delta y} \left(\rho^n_{i,j+1} - \rho^n_{i,j-1}\right) \right) +$$
$$+ \Delta t \left((\rho_s)_{i,j} - \rho^n_{i,j} \right) S^n_{i,j}$$

which is appropriate when there is predominance of convection, τ is almost zero.

We suppose also that aquifer is initially filled with freshwater [$\rho_o(x,y) = \rho_f$], and the initial velocities are

$$u_o(x,y) = 0.1$$
$$v_o(x,y) = 0.$$

The coefficients in equations (1) and (2) and the discretization parameters are taken to be:

$$\varepsilon = 0.1, \quad g = 10; \ \mu = 0.001$$
$$K = 0.001, \quad S = 0., 0.001, \ 0.01, \quad \tau = 0.001$$
$$\Delta x = 0.2, \quad \Delta y = 0.1, \quad \Delta t = 0.025$$

The figures 1, 2, 3, and 4 show the computed solutions at times t=5, t=10, t=20, and t=50, respectively, when S is zero, while the figures 5, 6, 7, 8 and 9, 10, 11, 12 show the computed solutions when S is, respectively, 0.001 and 0.01.

We notice that, in the first two examples the area which is influenced by the extraction well remains filled by freshwater only, while in the last example in such an area the freshwater is mixed with saltwater.

The numerical results that one obtains for time t > 50 do not differ substantially from those obtained at time t=50; thus, we can conclude that figures 4, 8 and 12 also represent the steady state solution for this problem.

(6)
$$\varepsilon_{i,j}\,\rho^n_{i,j} = \rho^n_{i,j} - \Delta t \Bigg[u^n_{i,j}\frac{1}{2\Delta x}(\rho^n_{i+1,j} - \rho^n_{i-1,j}) + v^n_{i,j}\frac{1}{2\Delta y}(\rho^n_{i,j+1} - \rho^n_{i,j-1}) - \tau\left(\frac{1}{(\Delta x)^2}(\rho^n_{i+1,j} - 2\rho^n_{i,j} + \rho^n_{i-1,j}) + \frac{1}{(\Delta y)^2}(\rho^n_{i,j+1} - 2\rho^n_{i,j} + \rho^n_{i,j-1})\right) - \left((\rho_S)_{i,j} - \rho^n_{i,j}\right) S^n_{i,j}\Bigg]$$

which is appropriate when convection and diffusion have the same order of magnitude, τ is strictly positive:

(7)
$$\varepsilon_{i,j}\rho^{n+1}_{i,j} = \rho^n_{i,j} - \Delta t \Bigg[\frac{u^n_{i,j}}{\Delta x}\begin{Bmatrix}\rho^n_{i,j} - \rho^n_{i-1,j}\\ \rho^n_{i+1,j} - \rho^n_{i,j}\end{Bmatrix} + \frac{v^n_{i,j}}{\Delta y}\begin{Bmatrix}\rho^n_{i,j} - \rho^n_{i,j-1}\\ \rho^n_{i,j+1} - \rho^n_{i,j}\end{Bmatrix} - \tau\left[\frac{1}{(\Delta x)^2}(\rho^n_{i+1,j} - 2\rho^n_{i,j} + \rho^n_{i-1,j}) + \frac{1}{(\Delta y)^2}(\rho^n_{i,j+1} - 2\rho^n_{i,j} + \rho^n_{i,j-1})\right] - \left((\rho_S)_{i,j} - \rho^n_{i,j}\right) S^n_{i,j}\Bigg],$$

which is appropriate for any non negative mass diffusivity τ.

In the equation (7) the upper or lower line of each brace is to apply as the corresponding coefficient $u^n_{i,j}$, $v^n_{i,j}$ is non negative or not. The discrete variables $u^n_{i,j}$ and $v^n_{i,j}$ are defined as simple averages from the closest scalar grid points, and the discrete source term $S^n_{i,j}$ at the center of each cell is defined as the integral of the source S over the whole area.

The implicit finite difference system corresponding to the system (2) is:

$$(8)\qquad \left\{ \begin{array}{l} \dfrac{\rho^{n+1}_{i+1/2,j}}{\varepsilon_{i+1/2,j}} \dfrac{u^{n+1}_{i+1/2,j} - u^n_{i+1/2,j}}{\Delta t} = - \dfrac{p^{n+1}_{i+1,j} - p^{n+1}_{i,j}}{\Delta x} - \dfrac{\mu}{K_{i+1/2,j}} u^{n+1}_{i+1/2,j} \\[2ex] \dfrac{\rho^{n+1}_{i,j+1/2}}{\varepsilon_{i,j+1/2}} \dfrac{v^{n+1}_{i,j+1/2} - v^n_{i,j+1/2}}{\Delta t} = - \dfrac{p^{n+1}_{i,j+1} - p^{n+1}_{i,j}}{\Delta y} - \dfrac{\mu}{K_{i,j+1/2}} v^{n+1}_{i,j+1/2} - \rho^{n+1}_{i,j+1/2} g \\[2ex] \dfrac{1}{\Delta x} (u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j}) + \dfrac{1}{\Delta y} (v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2}) = S^{n+1}_{i,j} , \end{array} \right.$$

vhere the coefficients $\rho^{n+1}_{i+1/2,j}$, and $\rho^{n+1}_{i,j+1/2}$ are obtained as simple averages from the values computed with equations (5) or (6) or (7).

The determination of unknowns $p^{n+1}_{i,j}$, $u^{n+1}_{i+1/2,j}$ and $v^{n+1}_{i,j+1/2}$ can be obtained by an iterative procedure but in practice the solution can be accomplished with much greater speed if one adjusts the velocities through changes in the pressure field at each iteration and if the iteration formula applied is that of successive overrelaxation. (For details see, e.g., [1], [2], [5], [7]).

A finite difference scheme which discretizes equation (3) and allows, simultaneously, for the correct physical boundary conditions uses a nonsymmetric discretization for the partial derivative of h with respect to the spatial variable.

If $h^n_{i+1/2}$ denotes the surface height defined on the right side of the i-th column of cells at time $t^0+n\Delta t$, the finite difference equation corresponding to equation (3) is taken to be

$$(9)\qquad \varepsilon \frac{1}{s\Delta t} (h^{n+1}_{i+1/2} - h^n_{i+1/2}) + \frac{u^n_s}{\Delta x} \left\{ \begin{array}{l} h^n_{i+1/2} - h^n_{i-1/2} \\ h^n_{i+\frac{3}{2}} \; - h^n_{i-1/2} \end{array} \right\} = v^n_s ,$$

where the upper or the lower line of the brace is to apply as the corresponding coefficients u^n_s is nonnegative or not.

The coefficients u^n_s and v^n_s are obtained as a weighted average from the nearest cell velocities.

We want to point out that the boundaries do not necessarily consist of horizontal or vertical lines. It is sufficient that the boundaries to be considered can be well approximated by piecewise linear arcs whose knots are on the grid lines.

Methods of dealing with normal derivative boundary condition for the equation (1), when the boundary is piecewise linear, can be found in [5]. If the pressure p_s is specified as a boundary condition for the system (2), the pressure $p_{i,j}^{n+1}$ at the boundary cell center, which is crossed by a boundary line segment, is chosen such that a linear interpolation between it and the pressure in the nearest interior cell yields the boundary pressure p_s^{n+1} (see, e.g., [1]) Clearly, the free surface is of this type when it exists.

Finally, when the normal velocity $q\perp$ is prescribed as boundary condition for the system (2) in a cell which is crossed by a boundary line segment, the basic idea which is to be implemented for all such cells is that a discrete incompressibility condition, like the third equation of (8), must be satisfied in each portion of a boundary cell which is part of mesh region. Thus the numerical method conserves rigorously the fluid volume in each cell and, in the case of a boundary cell, also in the portion which is in the interior of D. [3].

3. STABILITY CONSIDERATIONS

We refer to [1], [2] in order to have a rigorous stability analysis for the method described in this paper. Here we report only the restrictions for the time step Δt and the space steps Δx and Δy in order to get stability for the transport equation (1). Specifically, if (5) is chosen, it is sufficient to take:

$$(\Delta t)\max_{i,j}\left(\frac{|u_{i,j}^n|}{\Delta x}, \frac{|v_{i,j}^n|}{\Delta y}\right) \leq \frac{1}{2}$$

if (6) is chosen we get stability if the space steps satisfy:

$$(\Delta x)\max_{i,j}|u_{i,j}^n| \leq 2\tau$$

$$(\Delta y)\max_{i,j}|v_{i,j}^n| \leq 2\tau$$

and the time step satisfies:

$$\Delta t \leq \left[2\tau\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right)\right]^{-1};$$

finally if (7) is chosen this scheme is stable when:

$$(\Delta t)\max_{i,j}\left[\frac{|u_{i,j}^n|}{\Delta x} + \frac{|v_{i,j}^n|}{\Delta y} + 2\tau\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right)\right] \leq 1.$$

When the free surface calculation is included, the time step Δt has to satisfy the surface wave Courant condition

$$\Delta t \leq \frac{\Delta x}{\sqrt{g\, h_{max}}},$$

where h_{max} is the maximum fluid depth.

The numerical method for the system (8) is unconditionally stable [1] because it is implicit and linear.

4. COMPUTATIONAL EXAMPLES

Let us consider an homogeneous, isotropic aquifer D of length l=2 and thickness d=1, confined above and below by impermeable strata. A constant freshwater flux with velocity u(0,y,t) = 0.1 enters the aquifer along the vertical face x=0 and a constant saltwater is maintained at x=l (see, e.g., [1], [2]).

We assume also that in the aquifer there is a region R, defined by $0.2 \leq x \leq 0.4$ and $0.6 \leq y \leq 1$, which is influenced by an extraction well. The fluid flow in this problem is described by equations (1) and (2) with the following boundary conditions:

$$u(0,y,t) = 0.1$$

$$v(x,0,t) = v(x,d,t) = 0$$

$$p(l,y,t) = \rho_s g(d-y)$$

$$\rho(0,y,t) = \rho_f = 1$$

$$\rho(l,y,t) = \rho_s = 1.05$$

$$\left.\frac{\partial \rho}{\partial y}\right|_{\substack{y=0 \\ y=d}} = 0$$

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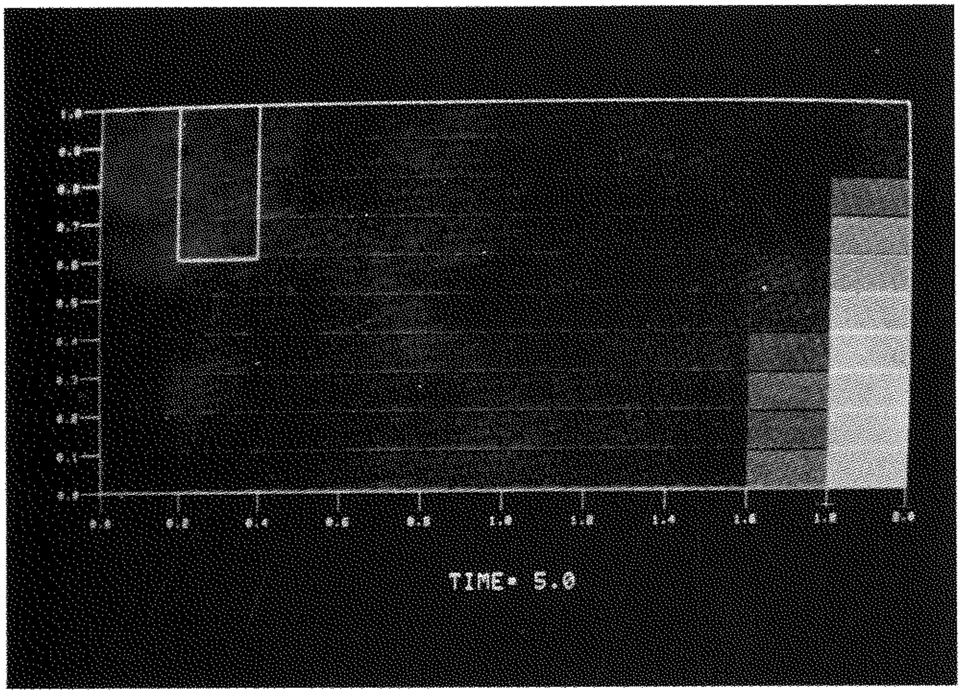


Fig. 1

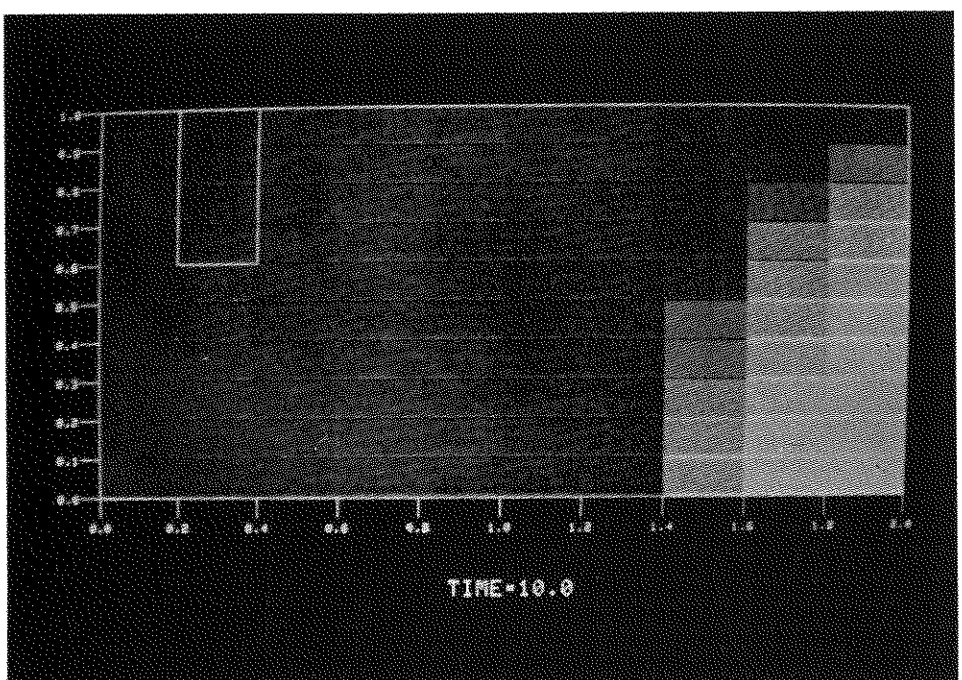


Fig. 2

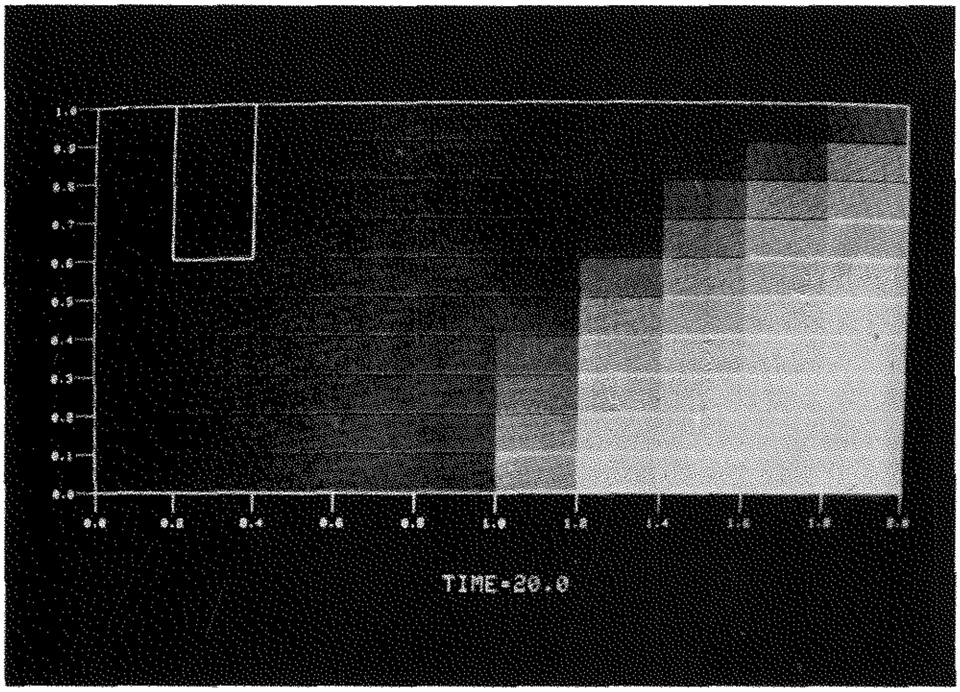
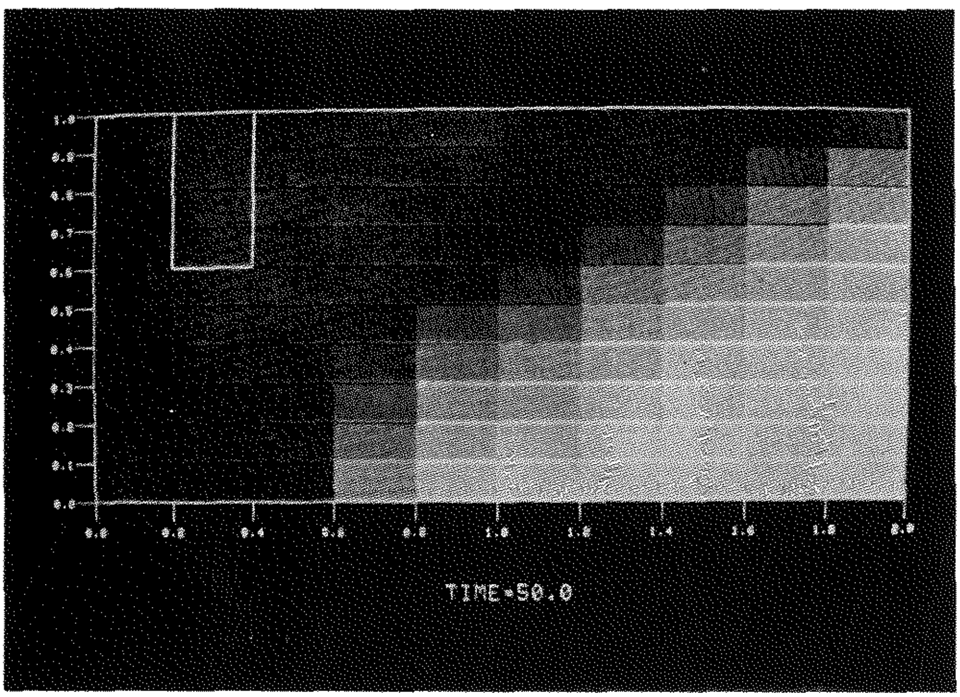


Fig. 3



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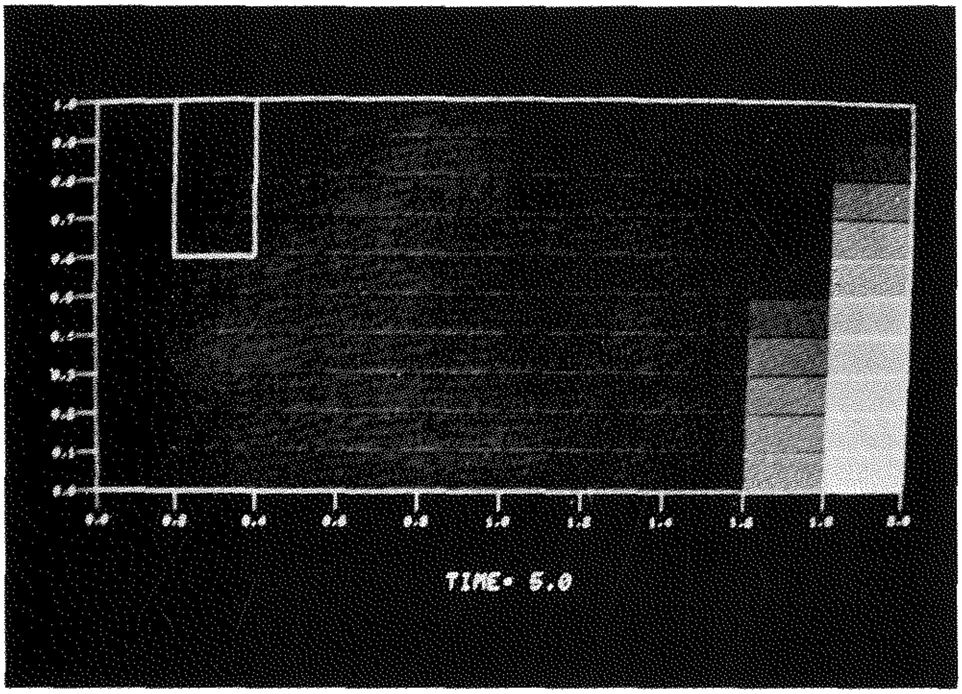
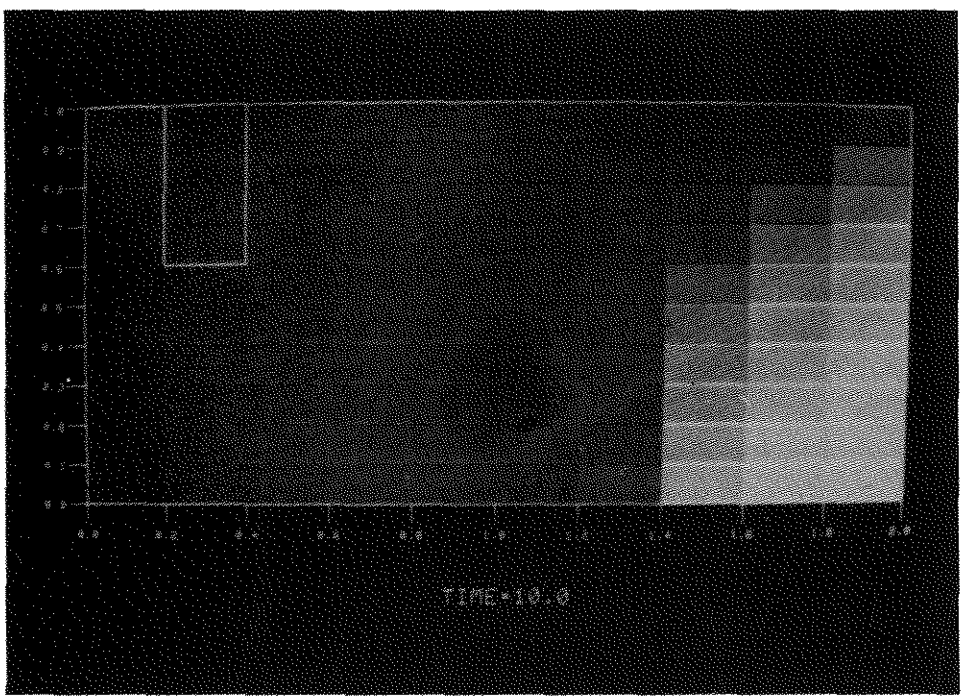


Fig. 5



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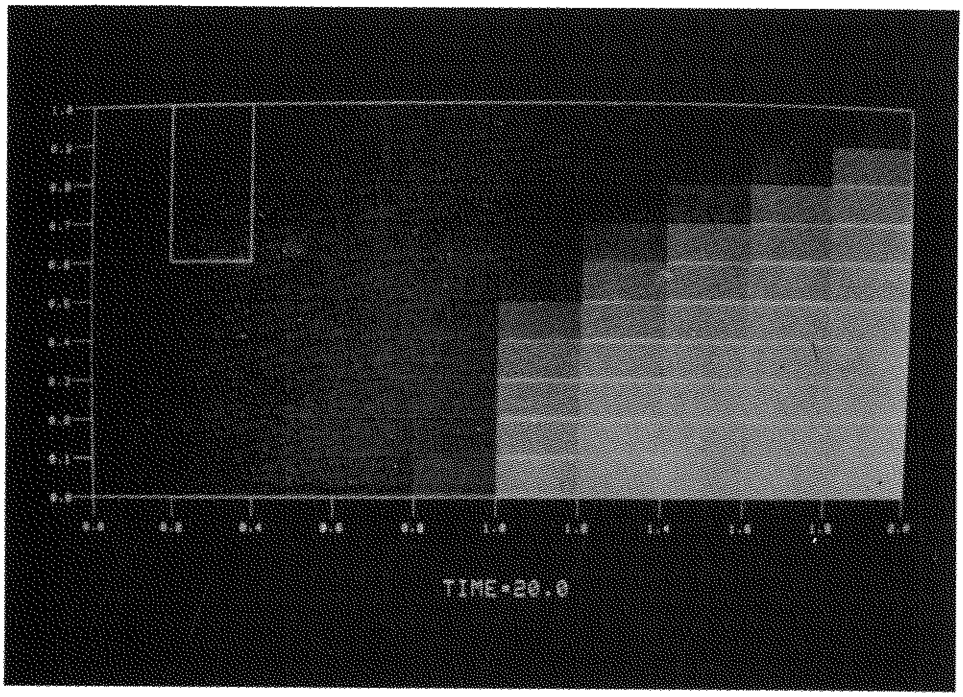
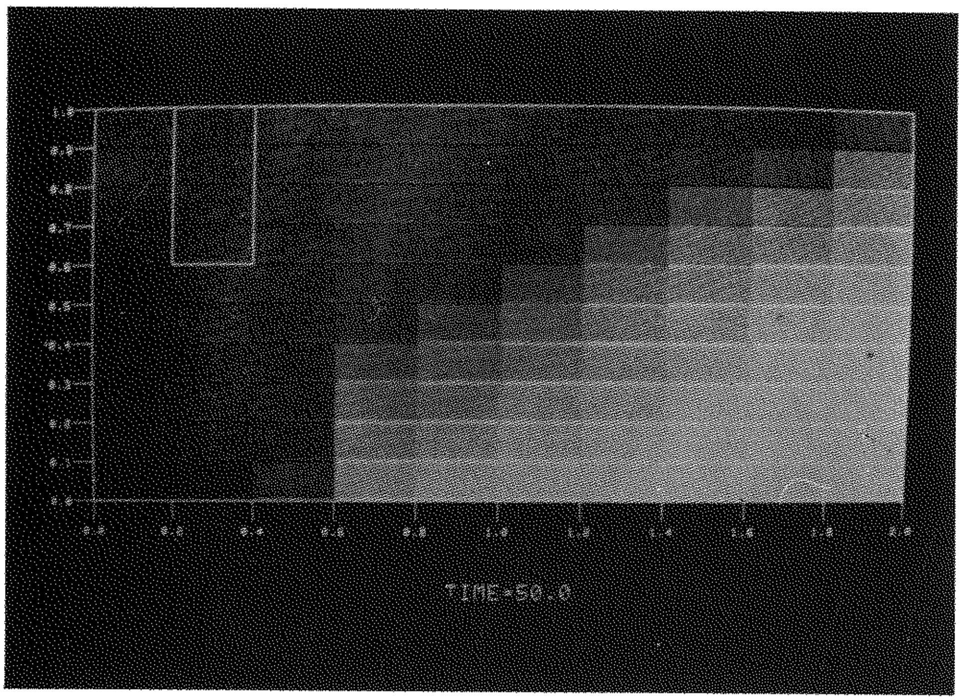


Fig. 7



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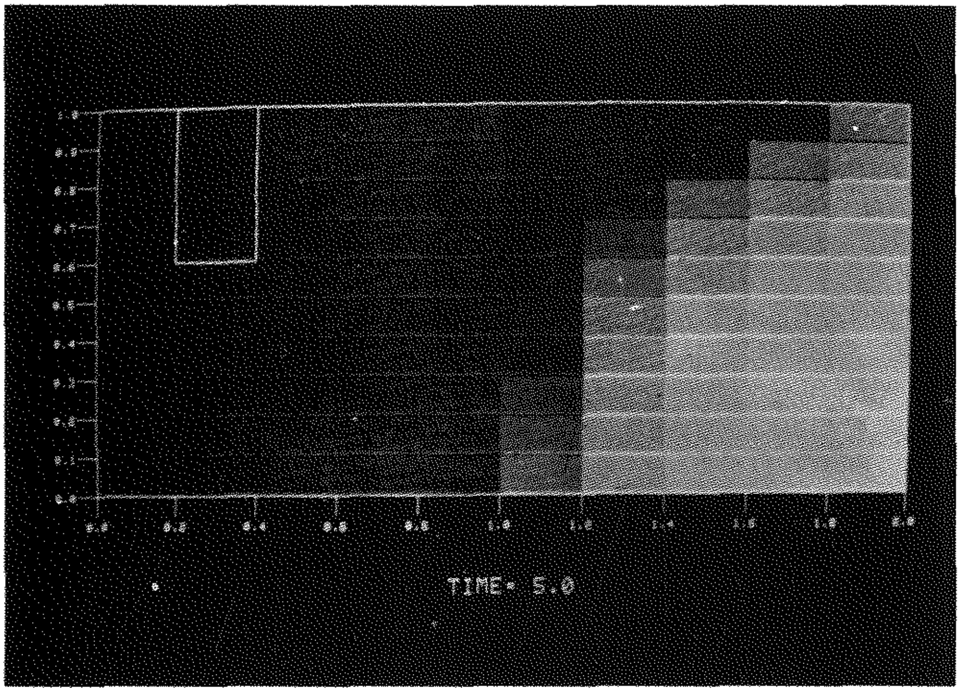

Fig. 9

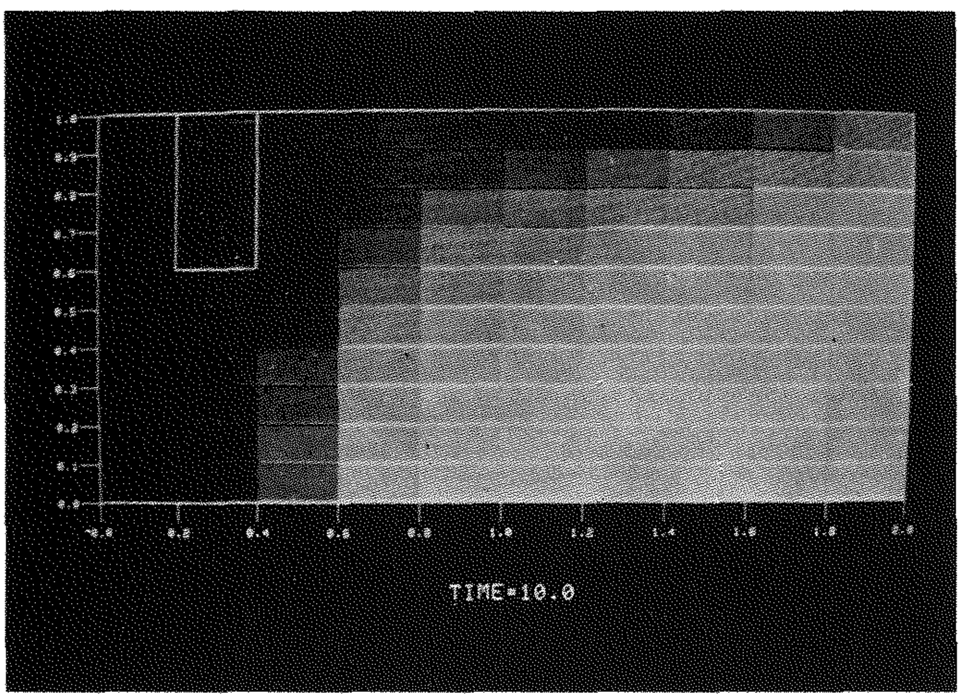

Fig. 10

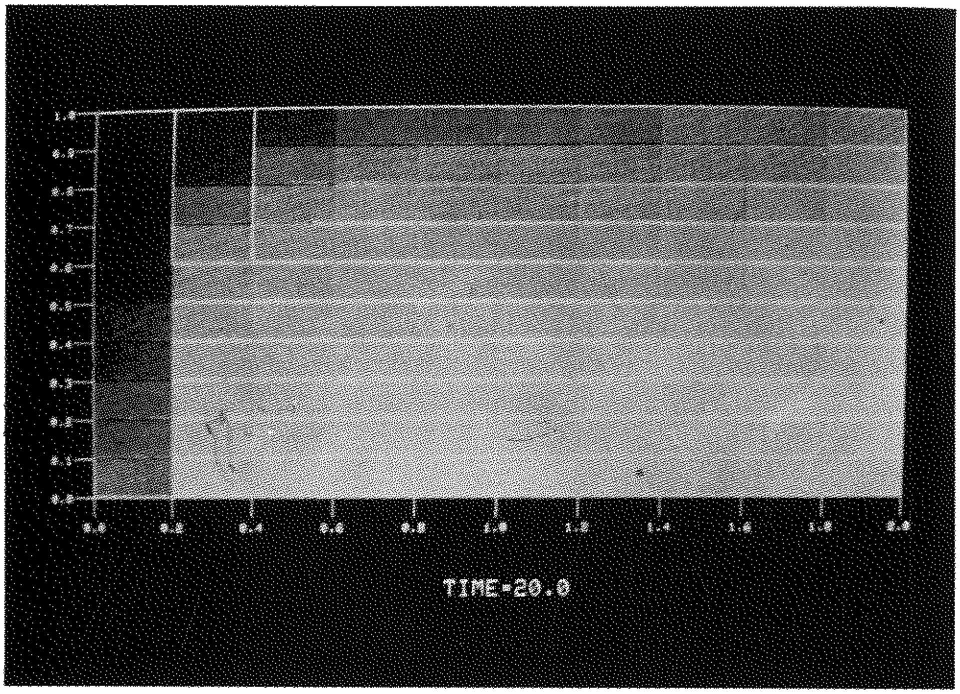

Fig. 11

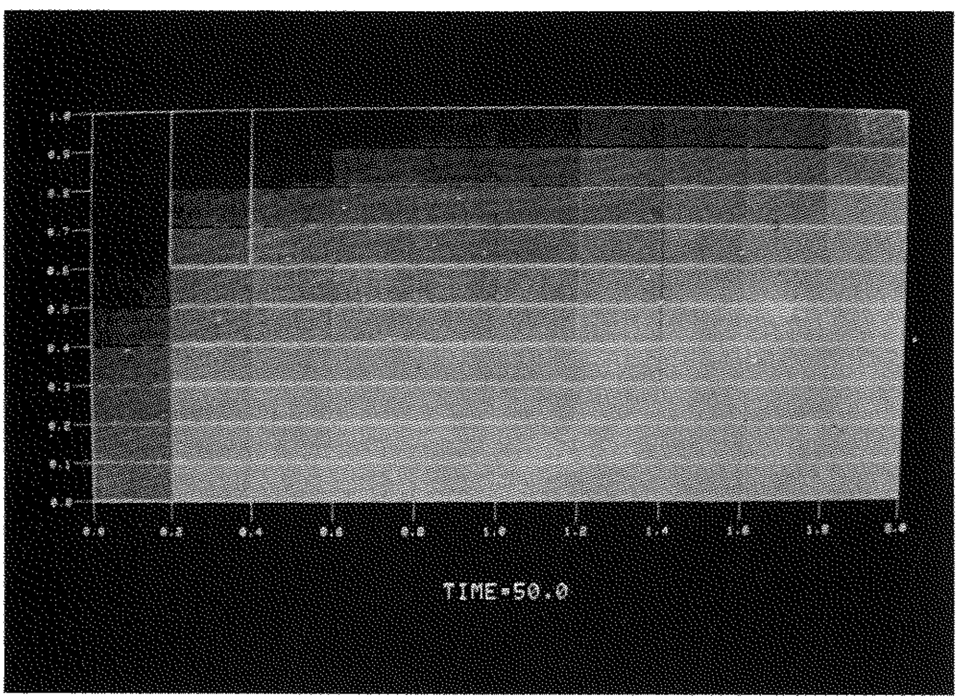

Fig. 12