G.A. BRUGGEMAN (*)

USE OF PRESSURE GENERATORS IN SOLVING THREE-DIMENSIONAL SALT-FRESH GROUND-WATER PROBLEMS

In this paper, a calculation method for groundwater flow with varying densities of the groundwater, for instance salt-fresh water flow in which the differences in density cannot be neglected, will be discussed. The method is, in a way, similar to the well known vortex theory where two-dimensional flow of an inhomogeneous fluid is treated by means of streamfunctions and vortices (De Josselin de Jong [2]; Van den Akker [1]) whereas this paper treats three-dimensional flow with the aid of pressure functions and pressure generators.

Considering confined or semi confined aquifers in coastal regions the following assumptions are made:

1. The soil is homogeneous and isotropic, so the intrinsic permeability is constant in all directions.

2. The viscosity μ of the fluid is constant.

3. The density ρ of the fluid is related to the concentration of a solute and is a function of place and time.

4. The soil skeleton and the fluid are incompressibile, so the elastic storage will be zero.

It follows from these assumptions that the density is not a function of pressure, but of the concentration of a solute.

Making use of a control volume having the form of a box with sides dx, dy and dz, the general continuity equation for the flux of groundwater that contains a solute, such as salt or chlorine, can be written as:

(*) National Institute for Water Supply, Voorburg, The Netherlands.

$$\text{div}\,(\rho\vec{v}) + \frac{\partial(\rho n)}{\partial t} = O \text{ or}$$

$$\frac{\partial}{\partial x}\,(\rho v_x) + \frac{\partial}{\partial y}\,(\rho v_y) + \frac{\partial}{\partial z}\,(\rho v_z) + \frac{\partial(\rho n)}{\partial t} = O \tag{1}$$

in which ρ = density, $\vec{v}$ = bulk velocity or Darcy velocity or specific discharge and n = porosity. We can evaluate these equations to

$$\rho \,\text{div}\, \vec{v} + \vec{v}.\,\text{grad}\,\rho + n\frac{\partial\rho}{\partial t} = 0 \tag{2}$$

bearing in mind that the soil was assumed incompressible (n = constant, so $\frac{\partial n}{\partial t} = 0$).

As the water is incompressible (assumption 4), an increase of the solute concentration in the control volume only results in a change of the density and not of the volume of the groundwater. Therefore it is allowed to apply the continuity equation for the volume, which means:

$$\text{div}\,\vec{v} = 0 \qquad \text{or} \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{3}$$

The continuity equation (2) thus can be divided into two equations, one for the groundwater and one for the solute:

$$\text{div}\,\vec{v} = 0$$
$$\text{and}\,\,\vec{v}.\,\text{grad}\,\rho + n\frac{\partial\rho}{\partial t} = 0 \tag{4}$$

The density $\rho = \rho\,(c)$ is a function of the concentration of the solute (chlorine content in salt-fresh water problems). So, as the density is not a constant, the general form of Darcy's law must be chosen for the equation of motion:

$$\vec{v} = -\frac{k}{\mu}\,(\text{grad}\,p + \rho g\,\text{grad}\,z)\text{ or}$$

$$v_x = -\frac{k}{\mu}\frac{\partial p}{\partial x}\quad v_y = -\frac{k}{\mu}\frac{\partial p}{\partial y}\quad v_z = -\frac{k}{\mu}\left(\frac{\partial p}{\partial z} + \gamma\right) \tag{5}$$

in which p = pressure of the groundwater, k = intrinsic permeability, g = gravity acceleration and $\gamma = \rho g$ = specific weight of the groundwater = $\gamma(c)$, a function of the concentration of the solute.

It has no sense, to introduce some kind of potential function instead of pressure in this case, because in density flow the velocity vector is no longer simply proportional to the gradient of a potential function: the flow is rotational and not potential.

Combining the equation of motion (5) with the two continuity equations (4) the following two differential equations for three dimensional density flow are found:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + \frac{\partial \gamma}{\partial z} = 0 \text{ and}$$

$$\frac{k}{\mu} \left(\frac{\partial p}{\partial x} \cdot \frac{\partial \gamma}{\partial x} + \frac{\partial p}{\partial y} \cdot \frac{\partial \gamma}{\partial y} + \frac{\partial p}{\partial z} \cdot \frac{\partial \gamma}{\partial z} + \gamma \frac{\partial \gamma}{\partial z} \right) = n \frac{\partial \gamma}{\partial t} \qquad (6)$$

$$\text{or in vector notation: } \nabla^2 p + \frac{\partial \gamma}{\partial z} = 0$$

$$\text{and } \frac{k}{\mu} \left(\text{grad } p \cdot \text{grad } \gamma + \gamma \frac{\partial \gamma}{\partial z} \right) = n \frac{\partial \gamma}{\partial t}$$

These two differential equations are mutually dependent and have to be solved simultaneously for $p(x, y, z, t)$ and $\gamma(x, y, z, t)$. Although both water and soil are assumed to be incompressible, *the flow nevertheless is non-steady* as a result of the varying density of the groundwater.

Only if $\vec{v} \cdot \text{grad } \rho = 0$, as can be seen from (4) the density flow may become steady, that if the streamlines everywhere are tangent to the surfaces $\rho = \text{constant}$.

Now the basic principle of the calculation method lies in the fact that it is possible to separate the original differential equation (2) in the two equations (6), of which the first is a steady one (no derivative with respect to time):

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + \frac{\partial \gamma}{\partial z} = 0 \qquad (7)$$

This is a differential equation of the Poisson type, which may be solved with the aid of a so called *singularity integral*, as follows:

At first the flow will be assumed to be a potential flow for which holds:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \qquad (8)$$

but where in any point where $\frac{\partial \gamma}{\partial z} \neq 0$, a so called « pressure discharge » is introduced with a strength that equals the value of $\frac{\partial \gamma}{\partial z}$ in that point.

Such a *pressure generator* in a point causes a small pressure change everywhere in the field, in the same way as an abstraction or injection of water in a point gives changes of head everywhere in the flow field.

That change of pressure is, besides on the strength of the generator, also dependent on the boundary values of the flow problem, expressed this time in pressures or pressure gradients. The total contribution of the pressure generators situated in all points of the field where $\dfrac{\partial \gamma}{\partial z} \neq 0$, to the value of the pressure p in an arbitrary point (x_0, y_0, z_0) of the field, then gives the solution of the first differential equation.

This means, that starting from an initial known distribution of the density or specific weight of the groundwater, the initial pressure distribution all over the field can be determined by solving the Poisson equation (7) with the aid of pressure generators, thus reducing nonpotential flow to potential flow. This may be done also for complicated three dimensional problems, as potential flow does not yield severe difficulties for a good computer.

As soon as the pressure distribution is known, also the velocity distribution, related to it by means of Darcy's law, can be determined.

Next consider a small time interval Δt. During that time the water particles will move according to the calculated velocity distribution.

Assuming that a certain particle always maintains the same solute concentration (dispersion is neglected; this will be discussed later) the density distribution will have been changed according to the displacement of the water particles, which can be calculated; computer programs are available for this purpose. With the new density distribution again the Poisson equation (7) can be solved and the corresponding new pressure distribution and also the related new velocity distribution can be determined.

In this way the non-steady density flow can iteratively be calculated, determining the concentration of a solute as a function of place and time as a result of any natural or human interference on the groundwater system.

Primarily the influence of *one pressure generator* with a constant intensity η on the pressures in an *infinite field* will be determined.

Consider a sphere with radius R in an infinite field.

Assume that the flow inside the sphere satisfies a Poisson equation of the form

$$\nabla^2 p_1 = \eta \ \text{ for } \ r \leq R \tag{9}$$

while outside the sphere a potential flow exists:

$$\nabla^2 p_2 = 0 \ \text{ for } \ r \leq R, \tag{10}$$

the problem thus being pure spherical.

Inside the sphere, $(r \leq R)$ using spherical coordinates (9) becomes:

$$\text{for } r \leq R: \quad \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dp_1}{dr} \right) = \eta \qquad (11)$$

with solution: $p_1 = \frac{1}{6} \eta r^2 - \frac{A}{r} + B$

with A and B arbitrary constants.

As $\frac{dp_1}{dr}$ (o) $= 0$ the constant A becomes zero, so

$$p_1 = \frac{1}{6} \eta r^2 + B \quad \text{and} \quad \frac{dp_1}{dr} = \frac{1}{3} \eta r$$

Outside the sphere $(r \geq R)$ equation (10) in spherical coordinates becomes:

$$\text{for } r \geq R: \quad \frac{d}{dr} (r^2 \frac{dp_2}{dr}) = 0$$

$$\text{with solution } p_2 = -\frac{C}{r} + D$$

The constant D vanishes, with the condition that for $r \to \infty$ p_2 becomes zero.

$$\text{So } p_2 = -\frac{C}{r} \text{ and } \frac{dp_2}{dr} = \frac{C}{r^2}$$

Continuity at the sphere surface requires that both $p_1(R) = p_2(R)$ and

$$\frac{dp_1}{dr}(R) = \frac{dp_2}{dr}(R)$$

The last condition gives $\frac{1}{3} \eta R = \frac{C}{R^2}$ from which $C = \frac{1}{3} \eta R^3$, while the first condition yields:

$$\frac{1}{6} \eta R^2 + B = -\frac{1}{3} \eta R^2 \text{ and thus } B = -\frac{1}{2} \eta R^2$$

The complete solution is:

$$r \leq R: \quad p_1 = -\frac{1}{6} \eta \; (3R^2 - r^2) \tag{12}$$

$$r \geq R: \quad p_2 = -\frac{1}{3} \eta \frac{R^3}{r} \tag{13}$$

According to (13), a spherical pressure generator with intensity η and strength $\eta \, V_R$, with $V_R = $ volume of the sphere $= \frac{4}{3} \pi R^3$ gives a change in pressure in an infinite field, which is inversely proportional to the distance from the centre of the sphere:

$$p = -\frac{\eta}{4\pi r} V_R \tag{14}$$

The dimension of $\eta = [FL^{-4}]$ and of strength $\eta V_R = [FL^{-1}]$.

The magnitude of the volume V_R in (14) is immaterial for the foregoing analysis; so the region of generator strength can be reduced to a volume as small as pleased, such that V_R becomes the infinitesimal small volume dV. The spherical pressure generator becomes a point generator that gives a small pressure change everywhere in the field according to:

$$dp = -\frac{\eta \; dV}{4\pi r} \tag{15}$$

If η is a function of place: $\eta = \eta(x,y,z)$ in some region R, then the total contribution of the infinite pressure generators in that region to the pressure in the field becomes:

$$p = -\frac{1}{4\pi} \iiint\limits_V \frac{\eta(x_0, y_0, z_0) \; dx_0 \, dy_0 \, dz_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

According to (7) is $\eta(x_0, y_0, z_0) = -\dfrac{\partial \gamma}{\partial z} (x_0, y_0, z_0)$

The solution of the Poisson differential equation (7) now can be found as the solution of the Laplace differential equation (8) with initial and boundary conditions for the same problem, assuming the groundwater homogeneous (constant γ) and add to it the pressure distribution, caused by an infinite number of infinitely small pressure generators with strength $-\dfrac{\partial \gamma}{\partial z}$ at those points, where the density varies in z-direction.

For an infinite field this pressure distribution becomes:

$$p_1 = \frac{1}{4\pi} \iiint_V \frac{1}{r} \frac{\partial \gamma}{\partial z} (x_0, y_0, z_0) \, dx_0, dy_0, dz_0 \tag{16}$$

$$\text{with } r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

in which the volume integral has to be taken over the region where $\dfrac{\partial y}{\partial z} \neq 0$.

As p_1 is the solution in an infinite field, a second solution p_2 must be found, such that $\nabla^2 p_2 = 0$ and $p_1 + p_2$ satisfies the initial and boundary conditions. The sum $p = p_1 + p_2$ then represents the pressure distribution caused by the pressure generators.

The integral in equation (16) is called a *singularity integral*, because of the fact that a rotational flow is transformed into a potential flow by means of introducing pressure generators in an infinite number of singular points.

A particular case of the singularity integral (16) originates from a *sharp interface* S between two fluids of different densities ρ_1 and ρ_2.

Assume for instance the equation of the interface initially $(t = 0)$.

$$z_i = z_i(x_i, \ y_i)$$

where i denotes interface. If we consider a strip between the two surfaces z_i and $z_i + dz_i$ then $\dfrac{\partial \gamma}{\partial z}$ becomes $\dfrac{\gamma_1 - \gamma_2}{dz_i}$ and the integration in (16) in z-direction becomes $\dfrac{\gamma_1 - \gamma_2}{r}$

where:

$$r = \sqrt{(x-x_i)^2 + (y-y_i) + |z-z_i(x_i, y_i)|^2}$$

So the singularity integral in an infinite field in the case of a sharp interface $z_i = z_i(x_i, \ y_i)$ become:

$$p_1 = \frac{\gamma_1 - \gamma_2}{4\pi} \iint_{S_{xy}} \frac{dx_i \ dy_i}{r} \tag{17}$$

with r defined above.

The integration must be performed over S_{xy}, the projection of the interface S on the xy-plane; r is the distance between a small element dS on the interface with projection $dx_i \, dy_i$ and the arbitrary point $P(x,y,z)$, where pressure changes take place in an infinite field as a result of an infinite number of pressure generators with intensity $\gamma_1 - \gamma_2$, located on the interface S.

Along an interface a so called *shear flow* exists, which means that there are differences in velocity, tangent to the interface, of the fluids with different density at both sides of the interface.

This can be shown as follows:

Assume that the interface seperates fresh water with specific weight γ_f from underlying salt water with specific weight γ_s.

Equilibrium considerations require, that at points of the interface, the pressure p_f and p_s for the fresh and the salt water are equal ($p_f = p_s = p_i$).

The piezometric heads at points of the interface become:

$$\emptyset_{if} = \frac{p_i}{\gamma_f} + z_i \quad \text{and} \quad \emptyset_{is} = \frac{p_i}{\gamma_s} + z_i$$

respectively. Elimination of p_i gives:

$$\gamma_f (\emptyset_{if} - z_i) = \gamma_s (\emptyset_{is} - z_i) \tag{18}$$

at the interface.

Differentiation of (18) with respect to x_i gives:

$$\gamma_f \left(\frac{\partial \emptyset_{if}}{\partial x_i} + \frac{\partial \emptyset_{if}}{\partial z_i} \cdot \frac{\partial z_i}{\partial x_i} - \frac{\partial z_i}{\partial x_i} \right) = \gamma_s \left(\frac{\partial \emptyset_{is}}{\partial x_i} + \frac{\partial \emptyset_{is}}{\partial z_i} \cdot \frac{\partial z_i}{\partial x_i} - \frac{\partial z_i}{\partial x_i} \right) \tag{19}$$

Within each region on both sides of the interface separately, Darcy's law for potential flow is applicable, however with different permeabilities:

$$v_{xf} = - K_f \frac{\partial \emptyset_f}{\partial x} = - \frac{k \gamma_f}{\mu} \frac{\partial \emptyset_f}{\partial x} \quad v_{xs} = - K_s \frac{\partial \emptyset_s}{\partial x} = \frac{k \gamma_s}{\mu} \frac{\partial \emptyset_s}{\partial x}$$

assuming $\mu_f = \mu_s = \mu$ (equal viscosities).

Thus at the interface we get with (19) a first relation between the differences of the Darcy velocities for the fluids on both sides of the interface:

$$(v_{xf} - V_{xs}) + (v_{zf} - v_{zs}) \frac{\partial z_i}{\partial x_i} = \frac{k}{\mu} (\gamma_s - \gamma_f) \frac{\partial z_i}{\partial x_i} \tag{20}$$

In the same way differentiating (18) with respect to y we find a second relation:

$$(v_{yf} - v_{ys}) + (v_{zf} - v_{zs}) \frac{\partial z_i}{\partial y_i} = \frac{k}{\mu} (\gamma_s - \gamma_f) \frac{\partial z_i}{\partial y_i} \tag{21}$$

The interface in general is non-steady and may be represented by $z_i = z_i (x_i, y_i, t)$.

At every point of this moving interface the following expression holds.

$$\frac{dz_i}{dt} = \frac{\partial z_i}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial z_i}{\partial y_i} \cdot \frac{dy_i}{dt} + \frac{\partial z_i}{\partial t} \qquad (22)$$

As $\dfrac{dx_i}{dt}$ represents both the real velocity component in x-direction for the fresh and the salt water, at points of the interface, and also $\dfrac{dy_i}{dt}$ and $\dfrac{dz_i}{dt}$ the real velocity components in y- and z-direction respectively, we get in terms of Darcy velocities:

$$\frac{\partial z_i}{\partial z_i} v_{xf} + \frac{\partial z_i}{\partial y_i} v_{yf} - v_{zf} + n_e \frac{\partial z_i}{\partial t} = 0$$

and

$$\frac{\partial z_i}{\partial x_i} v_{xs} + \frac{\partial z_i}{\partial y_i} v_{ys} - v_{zs} + n_e \frac{\partial z_i}{\partial t} = 0$$

Subtraction of these two equations gives the third relation between the differences of the Darcy velocities in the three coordinate directions on both sides of the interface at points of the interface:

$$(v_{xf} - v_{xs}) \frac{\partial z_i}{\partial x_i} + (v_{yf} - v_{ys}) \frac{\partial z_i}{\partial y_i} - (v_{zf} - v_{zs}) = 0 \qquad (23)$$

Denoting $v_{xf} - v_{xs}$ by d_x (d = difference) and $v_{yf} - v_{ys} = d_y$ and $v_{zf} - v_{zs} = d_z$ the equations may be solved for d_x, d_y and d_z:

$$d_x = v_{xf} - v_{xs} = \frac{k}{\mu} (\gamma_s - \gamma_f) \frac{\dfrac{\partial z_i}{\partial x}}{1 + (\dfrac{\partial z_i}{\partial x})^2 + (\dfrac{\partial z_i}{\partial y})^2}$$

$$d_y = v_{yf} - v_{ys} = \frac{k}{\mu}(\gamma_s - \gamma_f) \frac{\dfrac{\partial z_i}{\partial x}}{1 + (\dfrac{\partial z_i}{\partial x})^2 + (\dfrac{\partial z_i}{\partial y})^2} \tag{24}$$

$$d_z = v_{zf} - v_{zs} = \frac{k}{\mu}(\gamma_s - \gamma_f) \frac{(\dfrac{\partial z_i}{\partial x})^2 + (\dfrac{\partial z_i}{\partial y})^2}{1 + (\dfrac{\partial z_i}{\partial x})^2 + (\dfrac{\partial z_i}{\partial y})^2}$$

These velocity differences may be considered as the components of a vector $\vec{d}$ (d_x, d_y, d_z).

The vector $\vec{n}$, normal to the interface in a point at the interface, is, if we write the interface as:

$$f_i = z_i - z_i(x_i, y_i) = 0:$$

$$\vec{n} = \text{grad } f_i = (-\frac{\partial z_i}{\partial x}, -\frac{\partial z_i}{\partial y}, 1)$$

It can easily be seen that $\vec{d}$ and $\vec{n}$ are orthogonal vectors as the scalar product equals zero:

$$\vec{d} \cdot \vec{n} = 0$$

The absolute value of the velocity difference vector is, if $K^* = \dfrac{k}{\eta}(\gamma_s - \gamma_f)$:

$$|\vec{d}|^2 = d_x^2 + d_y^2 + d_z^2 = K^* \, d_z$$

as can be shown easily.

Conclusion: the velocity difference vector $\vec{d}$ at a point of an abrupt interface between two homogeneous fluids of different density is directed along the interface (lies in the plane tangent to the interface in that point) and is called *shear flow*; the absolute value amounts to:

$$|\vec{d}| = \sqrt{K^* d_z}$$

It follows that in the direction normal to the interface the velocity difference equals zero and the flow is continuous in that direction, as might be expected.

In two-dimensional flow, for example flow only in x- and z-direction, $\dfrac{\partial z}{\partial y} = 0$ and $\dfrac{\partial z}{\partial x} = \text{tg } \beta$, the equations (24) become:

$$d_x = K^* \sin \beta \cos \beta \text{ and } d_z = K^* \sin^2 \beta$$

and

$$|\vec{d}| = K^* \sin \beta$$

in which β = the angle between the tangent in a point of the interface line and the positive x-axis.

As soon as transport of a solute in groundwater takes place, as in the case of density flow, we encounter the phenomenon of *dispersion*.

However, the differential equations that describe the transport may be considered as additional to the here described differential equations and need not be neglected; instead of equations (6).

We get:

$$\nabla^2 p + \frac{\partial \gamma}{\partial z} = 0 \tag{26}$$

and $n \dfrac{\partial}{\partial_i}\left(D_{ij} \cdot \dfrac{\partial \rho}{\partial_j}\right) + \dfrac{k}{\mu}\left(\text{grad } p \cdot \text{grad } \rho + \rho g \dfrac{\partial \rho}{\partial z}\right) = n \dfrac{\partial \rho}{\partial t}$

if the density ρ is a linear function of the concentration of the solute. The dispersion terms are written according to the Einstein convention with i, j = x, y, z:

$$\frac{\partial}{\partial i}\left(D_{ij} \frac{\partial \rho}{\partial j}\right) = \frac{\partial}{\partial x}\left(D_{xx} \frac{\partial \rho}{\partial x} + D_{xy} \frac{\partial \rho}{\partial y} + D_{xz} \frac{\partial \rho}{\partial z}\right)$$

$$+ \frac{\partial}{\partial y}\left(D_{yx} \frac{\partial \rho}{\partial x} + D_{yy} \frac{\partial \rho}{\partial y} + D_{yz} \frac{\partial \rho}{\partial z}\right)$$

$$+ \frac{\partial}{\partial z}\left(D_{zx} \frac{\partial \rho}{\partial x} + D_{zy} \frac{\partial \rho}{\partial y} + D_{zz} \frac{\partial \rho}{\partial z}\right)$$

G. A. BRUGGEMAN

REFERENCES

1 – VAN DEN AKKER, C., *Numerical analysis of the stream function in plane groundwater flow.* Thesis Technical University, Delft, Holland, 1982.

2 – DE JOSSELIN DE JONG, G., *Review of vortex theory for multiple fluid flow.* Delft Progress Report 2, 225-236, 1977.