DEVELOPMENT OF FRESH WATER LENSES

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ABSTRACT

Fresh water lenses in small islands surrounded by a sea or an ocean develop as a result of natural recharge. They have an important function as a resource for domestic and agricultural water supply.

In this paper approximative analytical solutions for the development of a fresh water lens in case of an elongated island (1-dimensional situation) and a circular island (axial-symmetric situation) are reviewed. An interesting and determining quantity appears to be the so-called time constant, which may easily be determined from the characteristics of the area.

INTRODUCTION

Water supply on small islands depends mainly on the amount of rainfall. Rain is partly captured and may directly satisfy the demand, the major part occurs as surface run-off and/or infiltrates into the subsurface.
The storage of groundwater by natural recharge is very often the main source for water supply, surface water is on most small islands of minor importance. In general these fresh water bodies appear as lenses, due to the density difference between the infiltrated water (fresh) and the saline groundwater.
In the next sections the development of fresh water lenses is described for two simple and strongly schematised cases, assuming a sharp interface between fresh and saline groundwater, saline water is at rest and there is only horizontal flow in the fresh water zone. Moreover, the subsoil is assumed to be homogeneous and isotropic.

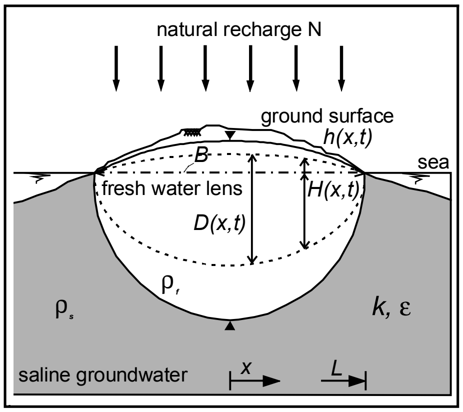
FRESH WATER LENS IN AN ELONGATED ISLAND

Applying the mass balance, Darcy's Law and the Badon-Ghijben Herzberg relation, in case of a long island, width *2L* and surrounded by a sea or an ocean (Figure 1), results in the following differential equation for the groundwater table:

$$2\frac{\varepsilon}{k}\frac{\partial\, h(x,t)}{\partial\, t} - \frac{\partial^2 h^2(x,t)}{\partial\, x^2} - \frac{2N}{k\,(1+\alpha)} = 0 \qquad (1)$$

$h(x, t)$ = groundwater table above m.s.l. [m]
$H(x,t)$ = depth fresh/saline interface below m.s.l. [m]; $H(x,t)= \alpha\, h(x,t)$
$D(x,t)$ = thickness of the fresh water lens [m]; $D(x,t)=(1+\alpha)h(x,t)$
k = permeability [m/day]

ε = *porosity [-]*
x = *distance from the centre [m]*
N = *natural recharge [m/day]*
t = *time [day]*
ρ_f = *specific mass fresh water [kg/m³]*
ρ_s = *specific mass saline water [kg/m³]*



α = $\rho_f/(\rho_s - \rho_f)$ [-]

Figure 1 Fresh water lens in an elongated island

The differential equation is non-linear, to find a solution, the shape of the lens during development is assumed to be conform the shape in steady state, however multiplied by a certain time function $F_l(t)$ (Brakel,1968). From this function is already known: $F_l(t)= 0$ for t=0 and $F_l(t)= 1$, for t=∞.

The groundwater table in steady state can be easily determined by neglecting the storage term of equation 1:

$$h(x,\infty) = [\frac{N(L^2 - x^2)}{k(1+\alpha)}]^{1/2} \quad (2)$$

$$h(x,\infty) = p_l (L^2 - x^2)^{1/2}, \quad p_l = [\frac{N}{k(1+\alpha)}]^{1/2} \quad (3)$$

The non-steady state of the groundwater table can then be described as:

$$\begin{aligned} h(x,t) &= F_l(t) * h(x,\infty) \\ &= F_l(t) * p_l \, (L^2 - x^2)^{1/2} \end{aligned} \quad (4)$$

In order to find a suitable function for $F_l(t)$, the solution for $h(x,t)$ is inserted in the differential equation and integrated over (half) the width of the island. The formula for $F_l(t)$ derived in this way, results in a solution for $h(x,t)$, which always satisfies the differential equation on average over the length of the cross-section.

The different parts of differential equation (1) are now elaborated. $F_l(t)$ and its derivative in time are indicated as F_l and F_l' respectively.

$$\frac{\partial h(x,t)}{\partial t} = F'_l \, p_l (L^2 - x^2)^{1/2} \quad (5)$$

$$\frac{\partial (h^2(x,t))}{\partial x} = \frac{\partial}{\partial x}[F_l^2 \, p_l^2 (L^2 - x^2)] = -2 F_l^2 \, p_l^2 \, x$$

$$\frac{\partial^2 (h^2(x,t))}{\partial x^2} = -2 F_l^2 \, p_l^2 \quad (6)$$

Substitution of (5) and (6) in (1), results in the following expression:

$$\frac{\varepsilon}{k} F'_l \, p_l (L^2 - x^2)^{1/2} + F_l^2 \, p_l^2 - p_l^2 = 0 \quad (7)$$

Integration of (7) from $x=0$ tot $x= L$:

$$\int_0^L [\frac{\varepsilon}{k} F'_l \, p_l (L^2 - x^2)^{1/2} + F_l^2 \, p_l^2 - p_l^2] \, dx = 0$$

$$\frac{\pi}{4}\frac{\varepsilon}{k} F'_l \, p_l L^2 + (F_l^2 - 1) \, p_l^2 L = 0 \quad (8)$$

Rewriting equation (8):

$$\frac{F'_l}{1 - F_l^2} = \frac{4 \, p_l k}{\pi \varepsilon L} \quad (9)$$

The time function $F_l(t)$, which satisfies this equation is:

$$F_l(t) = \tanh \frac{4 \, p_l k}{\pi \varepsilon L} t = \tanh(t/\tau_l) \ \ [-] \quad (10)$$

The integration constant is 0, because for $t= 0$, $F_l(t)= 0$.
τ_l in equation (10) is a time constant:

$$\tau_l = \frac{\pi \varepsilon L}{4} \left(\frac{1+\alpha}{k N}\right)^{1/2} \ \ [d] \quad (11)$$

τ_l is a determining factor, for the way a system is responding to an input signal. τ_l is determined by (geo)-hydrologic characteristics of the area.

The position of the groundwater table related to m.s.l. with (4) and (10) is:

$$h(x,t) = p_l (L^2 - x^2)^{1/2} * \tanh(t/\tau_l) \quad [m] \qquad (12)$$

At $t=\tau_l$, $h(x,t)$ is already at 76 % of its maximum value.
The maximum value is almost (99.5 %).achieved at $t=3\tau_l$.
The thickness of the lens $(D_l(x,t))$ can simply be derived by multiplying (12) by a factor $(1+\alpha)$.

As stated before, the solution satisfies the differential equation at each point in time, *on average* over the area (so not necessarily everywhere and always), so one has to be careful with formulae derived from this solution. As an example, the growth rate of the lens, the time derivative of $D_l(x,t)$:

$$\frac{\partial D_l(x,t)}{\partial t} = \frac{\partial}{\partial t}[(1+\alpha)\ p_l\ (L^2 - x^2)^{1/2} \tanh(t/\tau_l)]$$
$$= \frac{4}{\pi}\frac{N}{\varepsilon}(1-\frac{x^2}{L^2})^{1/2} * \cosh^{-2}(t/\tau_l) \quad [m/d] \qquad (13)$$

The formula shows clearly the growth rate at $t=0$ and $x=0$ is $>N/\varepsilon$, viz. $4N/\pi\varepsilon$.
The growth rate of the lens is depicted in Figure 2. Initially and for a great part of the area, the solution leads to a growth rate more than N/ε. For $t \geq 0{,}5\tau_l$ the situation becomes more realistic.
The outflow into the sea $(x=L)$ -actually that part of the recharge that is lost- is found by applying Darcy's Law:

$$q(x,t) = -k[(1+\alpha)\ p_l (L^2 - x^2)^{1/2} * \tanh(t/\tau)]\frac{\partial h(x,t)}{\partial x}$$
$$= N x * \tanh^2(t/\tau) \quad [m^2/d] \qquad (14)$$

Important, for abstraction of groundwater is the amount available in storage $(V_l(t))$. This quantity can be determined by integrating in time the natural recharge $(N.2L)$ minus the outflow into the sea $(q(L,t))$:

$$V_l(t) = 2\int_0^t [N L - NL \tanh^2(t/\tau_l)]\, dt$$
$$= 2 N L \tau_l * \tanh(t/\tau_l) \quad [m^3/m] \qquad (15)$$

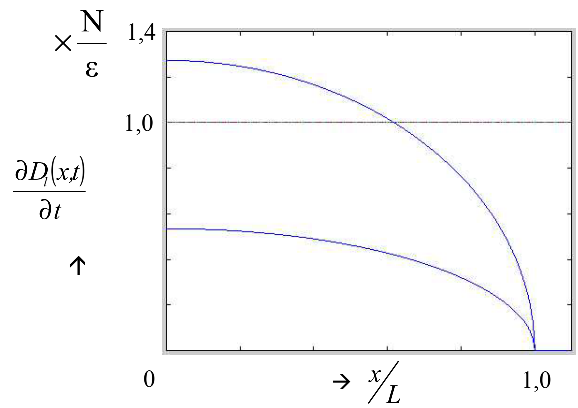


Figure 2 Growth rate of a fresh water lens in an elongated island

The maximum storage is achieved at $t>3\tau_l$, then $\tanh(t/\tau_l) \approx 1$:

$$V_{l\,max.} = 2 N L \tau_l \quad [m^3/m] \qquad (16)$$

The maximum amount of water in storage appears to be the product of the time constant τ in [d] and the natural recharge $N.2L$ in $[m^3 m^{-1} d^{-1}]$.

Fresh water lens in a circular island

The differential equation for the groundwater table due to natural recharge, in case of a circular island surrounded by a sea or ocean -an axial symmetric situation- can be formulated as follows (Figure 3):

$$2\frac{\varepsilon}{k}\frac{\partial h(r,t)}{\partial t} - \frac{\partial^2 h^2(r,t)}{\partial r^2} - \frac{1}{r}\frac{\partial h^2(r,t)}{\partial r} - \frac{2N}{k(1+\alpha)} = 0 \qquad (17)$$

h(r,t) = groundwater table above m.s.l. [m]

H(r,t) = depth of fresh/saline interface below m.s.l. [m], H(r,t)= $\alpha\, h(r,t)$
D(r,t) = thickness of fresh water lens [m],
D(r,t)=(1+α)h(r,t)
(All other symbols as indicated before)

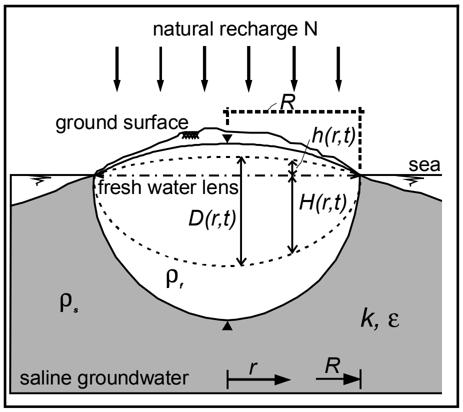


Figure 3 Fresh water lens in a circular island

Again the differential equation is non-linear. We assume also in this case a solution conform the steady state shape of the fresh water lens, but multiplied by a time function $F_o(t)$ (Boekelman & Grakist, 1972). Although this function is not known yet, we know its value at *t= 0:* $F_o(t)= 0$ and at t= ∞: $F_o(t)= 1$.

The solution of the groundwater table in steady state is:

$$h^2(r,\infty) = \frac{N\,(R^2 - r^2)}{2\,k(1+\alpha)} \qquad (18)$$

$$h(r,\infty) = p_o\,(R^2 - r^2)^{1/2},\ met\ \ p_o = [\frac{N}{2k\,(1+\alpha)}]^{1/2} \qquad (19)$$

So in non-steady state:

$$h(r,t) = F_o(t) * h(r,\infty) = F_o(t) * p_o\,(R^2 - r^2)^{1/2} \qquad (20)$$

Substitution of *h(r,t)* in equation (17) and integration over the whole area will result in such a solution for $F_o(t)$, that the differential equation is always satisfied *on average* over the area.

For simplicity, the different terms are elaborated one after the other. $F_o(t)$ and its derivative in time are indicated as F_o and F_o' respectively.

$$\frac{\partial\, h(r,t)}{\partial\, t} = F'_o * p_o\,(r^2 - r^2)^{1/2}$$

(21) $$\frac{\partial\, h^2(r,t)}{\partial\, r} = \frac{\partial}{\partial\, r}\,[\,F_o^{\,2} * p_o^2\,(R^2 - r^2)] = -2\,r\ \ p_o^2\,F_o^{\,2}$$

(22)

$$\frac{\partial^2\, h^2(r,t)}{\partial\, r^2} = -2\ \ p_o^2\,F_o^{\,2} \qquad (23)$$

Substitution of (21), (22) and (23) in (17) results in the next expression:

$$\frac{\varepsilon}{k}\,F'_o\,(R^2 - r^2)^{1/2} + 2\ \ p_o\,(F_o^{\,2} - 1) = 0 \qquad (24)$$

Integration of (24) over the area:

$$\int_0^R [\frac{\varepsilon}{k}\,F'_o\,(R^2 - r^2)^{1/2} + 2\,p_o\,(F_o^{\,2} - 1)]\ r\,dr = 0 \qquad (25)$$

This leads to:

$$\frac{F'_o}{(1 - F_o^{\,2})} = \frac{3\,k\,p_o}{\varepsilon\,R}\ , \qquad (26)$$

and a solution for the time function:

$$F_o = \tanh(\frac{3\,k\,p_o}{\varepsilon\,R}\,t) = \tanh(t/\tau_o)\quad [-] \qquad (27)$$

The integration constant is *0*, as $F_o= 0$ for *t= 0*. τ_0 in equation (26) is again a time constant and comprises the characteristics of the area.

$$\tau_o = \frac{\sqrt{2}}{3}\ \varepsilon\,R\ (\frac{1+\alpha}{k\,N})^{1/2}\quad [d] \qquad (28)$$

Combining (20) and (27), delivers a formula for the groundwater table with respect to m.s.l.:

$$h(r,t) = p_o\,(R^2 - r^2)^{1/2} * \tanh(t/\tau_o)\quad [m] \qquad (29)$$

The thickness of the lens *(D(r,t))* is found by multiplying (29) by *(1+α)*, after which the growth rate of the lens can be derived easily. Again this delivers too high values in the initial phase of the development of the lens.

The flow in the lens is:

$$Q(r,t) = -2\pi r k .(1+\alpha)\, p_o (R^2 - r^2)^{1/2} \tanh(t/\tau_o) * \frac{\partial h(r,t)}{\partial r} \quad [m^3 d^{-1}]$$

This yields:

$$Q(r,t) = N\pi r^2 * \tanh^2(t/\tau_o) \quad [m^3 d^{-1}] \qquad (30)$$

For $r = R$, we find the outflow into the sea. At $t = \tau_0$, the outflow is 58%, at $t = 3\tau_0$ 99 % of the natural recharge.

The amount of fresh water in storage can be determined from the water balance:

$$V(t) = \int_0^t [N\pi R^2 - N\pi R^2 * \tanh^2(t/\tau_o)]\, dt$$
$$= N\pi R^2 t - N\pi R^2 [t - \tau_o \tanh(t/\tau_o)] \qquad (31)$$
$$= N\pi R^2 \tau_o \tanh(t/\tau_o) \quad [m^3]$$

The maximum storage:

$$V_{max.} = N\pi R^2 \tau_o \quad [m^3], \qquad (32)$$

is achieved in case $t \geq 3\tau_0$.

Again the maximum amount of fresh water in storage is the product of natural recharge (in $[m^3 d^{-1}]$) and time constant (in [d]).
The magnitude of the time constants (τ_l and τ_0.) and subsequent values of the hyperbolic functions involved, can be determined easily.

Figure 4 shows the relevant time functions.

CONCLUSIONS

The formulae as presented here give a quick insight into the development and magnitude of fresh water lenses and show clearly, how and to what degree the different characteristics affect the process of development. Especially the time constant is a determining factor.

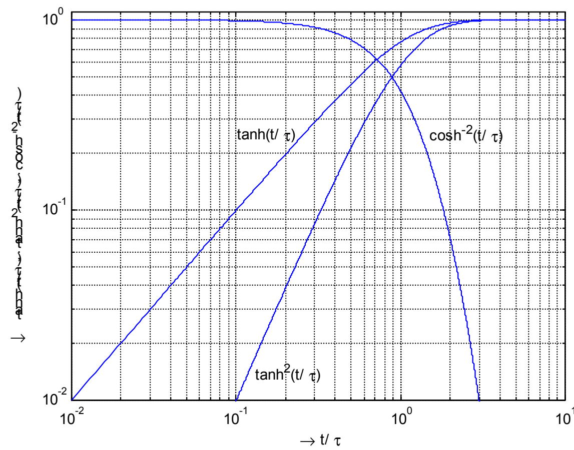


Figure 4 Functions tanh(x), tanh^2(x) and cosh^{-2}(x)

Examples

- ❑ Elongated island, with the following characteristics:
  N= 0,7 10^{-3} m/d
  L= 300 m
  k= 7,5 m/d
  ε= 0,35
  α= 50
  The time constant is:
  - *τ_l= 8087 days → = 22,2 [year].*
    After some 67 year *($\approx 3\tau_l$)* the fresh water lens approaches its steady state:
  - $D(0, 3\tau_l) \approx 17{,}80$ [m]
  - $V_l(3\tau_l) \approx 2942$ $[m^3.m^{-1}]$

- ❑ A circular island, applying the same data, with R=300 m:
  - *τ_o= 4878 days → = 13,4 [year].*
    After 40 years *($\approx 3\tau_o$)* the steady state situation is approached:
  - *$D(0, 3\tau_o) \approx 14{,}56$ [m]*
  - *$V_o(3\tau_o) \approx 960600$ $[m^3]$.*

REFERENCES

BRAKEL, J. (1968) The development in time of a fresh water lens by natural recharge in a dune area (in Dutch). Part of MSc-thesis, Delft University of Technology.

BOEKELMAN, R.H.& GRAKIST. G. (1972). Determination of shape and thickness of the fresh water lens in the Haringvreter. (in Dutch). Part of MSc-Thesis, Delft University of Technology.