Transient simulation of a real 2-D aquifer by network method

A. Soto Meca, F. Alhama and C. F. González-Fernández

Abstract The network simulation method is here applied to numerically solve transient contours of isochlor and stream function lines, as well as to locate recirculation toes, in a simulation of the benchmark Henry problem and the 2 D real aquifer (Biscayne aquifer in Florida), whose data are taken from the bibliography. The set of coupled, partial, differential equations for pressure and salt concentration variables is based on the stream function formulation. The applied method goes beyond the scope of the classical analogy, which always refers to lineal equations, and has already been applied in this field to 1 D problems. The influence of the two main dimensionless parameters on the solution, the discharge parameter and the Peclet number, in the stability of the solution is studied. For the benchmark Henry problem, computed transient streamlines and isochlor contours are presented and successfully compared with those of other authors. Also, for the Biscayne aquifer, locations of the toes of saltwater recirculation, under various values of the Peclet number and a specified discharge parameter, are also shown.

Index Terms Density dependent flow and transport, Modeling, Network method, Numerical simulation, Saltwater intrusion

I. Introduction

RECENTLY, the stream function formulation has been used by many authors to solve density driven flow and solute transport problems [1]-[3]. In turn, some of these authors have created their own commercial codes [2]. The network simulation method [4], NSM hereinafter, is here applied to design a 2-D model and numerically solve this type of problem. It is applied both to simulate the Henry problem and the dynamic of the Biscayne aquifer in Florida, whose data are taken from the bibliography. The results of the first simulation are compared with the approximate analytical solution of the well studied Henry problem in order to demonstrate the versatility and accuracy of the present model. The results of the simulation of the Byscaine aquifer are also compared with those provided by other numerical method and can be used to further understanding of the sea water encroachment mechanism.

In salt intrusion non-steady problems, fluid density is dependent upon concentration, making the problem highly nonlinear. Salt water affects fluid density which, in turn, affects the local velocity field. The use of the stream function formulation presents significant advantages in numerical simulations involving very small hydraulic gradients because any solution of the flow fields in involving this formulation ensures to conservation of fluid mass. Velocities are easily calculated from the stream function.

Based on NSM, a network model is designed from the finite difference differential equations that result from the spatial discretization of the partial differential equations. NSM, which has been successfully used for numerical simulation of other types of nonlinear problem, such as moving boundary problems [12]-[15], makes use of the powerful capabilities of modern circuit simulation computer codes which use the most complex algorithms of calculus. Soto et al. [16] applied NSM to solve 1-D problems of refreshing processes in saline aquifers.Since two dependent variables exist, two independent (electrically isolated) circuits form the model. However, the total number of devices that are able to implement the terms of the discretized equations are three: a resistor, a capacitor or a controlled current source.

The same occurs when boundary conditions are implemented, the same three devices being used. In consequence, the design of the model is simple to program. In addition, since the theorems of Kirchhoff (current Kirchhoff law) are assumed by the circuit simulation code, the water and salt balances are inherent in the program, an advantage not directly implemented by other numerical methods. Pspice is the electrical simulation code chosen in this work [17].

The results of the simulation are graphically presented both by curves directly provided by Pspice and by streamlines and isochlor contours obtained by Matlab from the numerical data of Pspice.

The influence of the Peclet number - which causes numerical instabilities at large values, for a particular discharge parameter, is studied for a wide range of values of this parameters occurring in the Biscayne aquifer. Steady state contours and locations of the toes of saltwater recirculation are successfully compared with the solution of other authors.

Manuscript received September , 2006.

A. Soto Meca. Physics Applied Department, Polytechnic University of Cartagena. Campus Muralla de Mar. 30203 Cartagena Spain, e-mail: antonio.soto@upct.es)F. Alhama. Physics Applied Department, ETSII. UPCT, 30203 Cartagena Spain (phone: +34-968-325512; fax: +34-968-325337 e-mail: paco.alhama@upct.es). C. F. González-Fernández. Physics Applied Department, Polytechnic University of Cartagena. Campus Muralla de Mar. 30203 Cartagena Spain, e-mail: carlosf.gonzalez@upct.es)

II. NOMENCLATURE

a discharge parameter (dimensionless)
c salt concentration (kg/m^3)
C capacitor (F)
d height of the aquifer (m)
D diffusivity (m^2/s)
$\boldsymbol{D}$ dispersion coefficient (m^2/s)
g gravity acceleration (ms^{-2})
G controlled current source
j water and salt velocities (m/s)
l length of the aquifer (m)
M number of the volume element along y′
N number of the volume element along x′
p pressure (N/m^2)
Pe Peclet number (dimensionless), Pe = $Q/\varepsilon D$
q specific discharge and water velocity (m/s)
Q recharge per unit width (m^2/s)
R resistor (Ω)
S_s storavity ($kg^{-1}ms^{-2}$)
t time (s)
v velocity (m/s)
x, y spatial coordinates (m)
Δx, Δy width and height of the volume element (m)
ε porosity (dimensionless)
ξ aspect ratio of the aquifer (depth/width)
κ permeability of the porous medium (m^2)
γ relative density $\gamma=(\rho - \rho_o)/\rho_o = \Delta\rho/\rho_o$
$\boldsymbol{\mu}$ dynamic viscosity ($Nm^{-1}s^{-1}$)
ρ flow density (kg/m^3)
ψ stream function (m^2s^{-1})

Subscripts
c related to salt concentration
i i=1,2,…N
i,j centre of the volume element i,j. Also, volume element number
i± Δx' right and left ends (x coordinate) of the volume element
j j=1,2,.. M
j± Δy' upper and bottom ends (y coordinate) of the volume element
o related to freshwater
s related to saltwater
x, y related to spatial coordinates

Superscripts
' dimensionless quantity

III. THE GOVERNING EQUATIONS AND NETWORK MODEL

The set of equations that defines the mathematical model is the following:

$$\partial(\varepsilon\rho)/\partial t + \nabla\cdot(\varepsilon\rho\, \mathbf{v}) = 0 \quad (1)$$

$$\partial(\varepsilon\rho c)/\partial t + \nabla\cdot(\varepsilon\rho c\, \mathbf{v}) + \nabla\cdot[\varepsilon\rho c\, (\mathbf{v} - \mathbf{v}_c)] = 0 \quad (2)$$

$$\mathbf{q} = - (\boldsymbol{\kappa}/\mu)(\nabla p - \rho\mathbf{g}) \quad (3)$$

$$c\, (\mathbf{v} - \mathbf{v}_c) = - \boldsymbol{D}\cdot\nabla c \quad (4)$$

$$\rho = \rho_o(1 + \gamma c) \quad (5)$$

Equations (1) and (2) are the conservation laws for the fluid flow and solute transport, respectively; (3) is the Darcy law that relates seepage velocity with pressure; (4) is Fick´s law that represents the mixing of solute through the fluid by diffusion and dispersion; (5) is the lineal dependence of fluid density and concentration.

Rearranging these equations it is easy to obtain

$$\rho\, S_s\, (\partial p/\partial t) + \varepsilon\rho_o\gamma\, (\partial c/\partial t) - \nabla\cdot[(\boldsymbol{\kappa}\rho/\mu)(\nabla p - \rho\mathbf{g})] = 0 \quad (6)$$

$$\varepsilon\rho\, (\partial c/\partial t) - \rho[(\boldsymbol{\kappa}/\mu)(\nabla p - \rho\mathbf{g})] \cdot\nabla c - \nabla\cdot(\varepsilon\rho\, \boldsymbol{D}\cdot\nabla c) = 0 \quad (7)$$

These equations, in terms of the stream function ψ (related to water velocity by $\partial\psi/\partial x = qy$ and $\partial\psi/\partial y = -qx$) and using the Boussisnesq approximation hypothesis, $\rho \cong \rho o$ (which implies $\nabla\cdot q = 0$), can be written as

$$\partial[(\mu/k)(\partial\psi/\partial x)]/\partial x + \partial[(\mu/k)(\partial\psi/\partial z)]/\partial z = -g\rho_o\gamma\, (\partial c/\partial x) \quad (8)$$

$$\varepsilon\, (\partial c/\partial t) + (\partial\psi/\partial y)(\partial c/\partial x) - (\partial\psi/\partial x)(\partial c/\partial y) - \varepsilon D(\Delta c) = 0 \quad (9)$$

Now, using the new variables $x' = x/l$, $y' = y/d$, $t' = t(D/d2)$, $qx' = - \partial\psi'/\partial y' = qx(d/Q)$, $qy' = \partial\psi'/\partial x' = qy(d/Q)$, and $\rho' = (\rho - \rho o)/(\rho s - \rho o)$, (8) and (9) can finally be written in dimensionless form as

$$(\partial^2\psi'/\partial x'^2) + (\partial^2\psi'/\partial y'^2) = (1/a)\, (\partial c'/\partial x') \quad (10)$$

$$(\partial^2 c'/\partial x'^2) + (\partial^2 c'/\partial y'^2) - Pe\, (\partial\psi'/\partial x')(\partial c'/\partial y') +$$

$$+ Pe(\partial\psi'/\partial y')(\partial c'/\partial x') = (\partial c'/\partial t') \quad (11)$$

$$a = \mu Q/(\kappa g d(\rho_s - \rho_o))$$

and

$$Pe = Q/(\varepsilon D)$$

are the discharge parameter and the Peclet number.

The equations of the boundary conditions depend on each aquifer. For example, for the benchmark Henry problem, figure 1, the boundary conditions expressed as a function of the dimensionless variables, for ξ=2, are

$$c'(0,y') = 0, \quad c'(2,y') = 1 \quad (12)$$

$$\partial c'(x',0)/\partial y' = \partial c'(x',1)/\partial y' = 0 \quad (13)$$

$\psi'(x',0) = 0,\ \ \psi'(x',1) = 1$
(14)

$\partial\psi'(0,y')/\partial x' = \partial\psi'(2,y')/\partial x' = 0$ (15)

The initial condition is given by

$$c'(x',y',t=0) = 0 \qquad (16)$$

Details for designing the network model from equations (10) and (11) are given in [4]. Using the nomenclature of figure 2 for the ends of a typical volume element, the network model is shown in figure 3.

The resistors (R) and capacitor (C) represent the lineal terms of the spatially discretized equations, while controlled current sources (G) implement nonlineal terms. Figure 3 represents the network model of the volume element which is formed by two electrically independent circuits. The coupling between equations is made through the controlled sources, whose output (dependent on the variables stream function and salt concentration) is specified in the model by the software. The network model of the volume element is 2-D interconnected (N×M volume elements) and boundary conditions added at the extremes by simple devices complete the model.

Once the network model is completed, its simulation is carried out in Pspice [16] with no other mathematical manipulation, simultaneously providing the solution for ψ' and c', both in tabulated and graphic forms.

The handling of this code for idealized situations is easy and "user-friendly". Since very few devices make up the network, very few programming rules are needed. Besides, the possibility exists of designing the network by sketching, using the option "schematics" generally included in the software of circuit simulation.

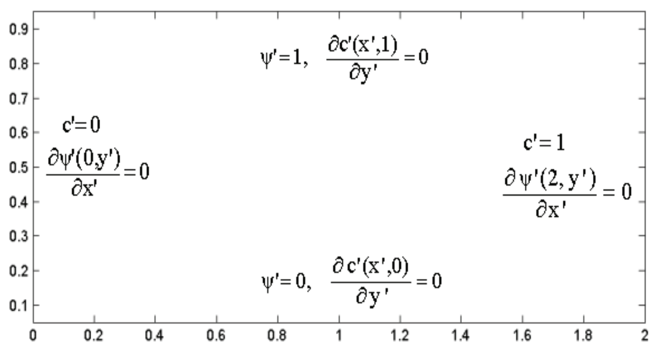

Figure 1. Boundary conditions for Henry problem.

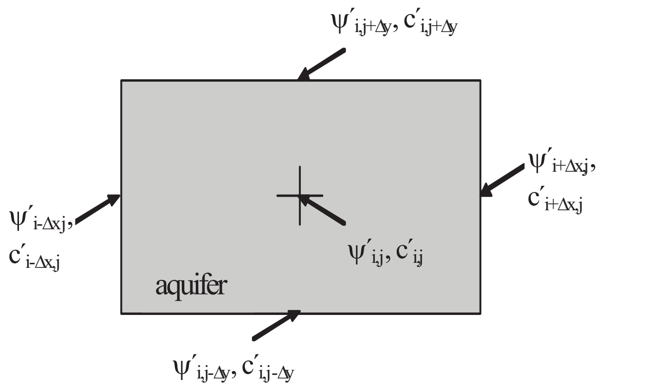

Figure 2. Nomenclature of the volume element.

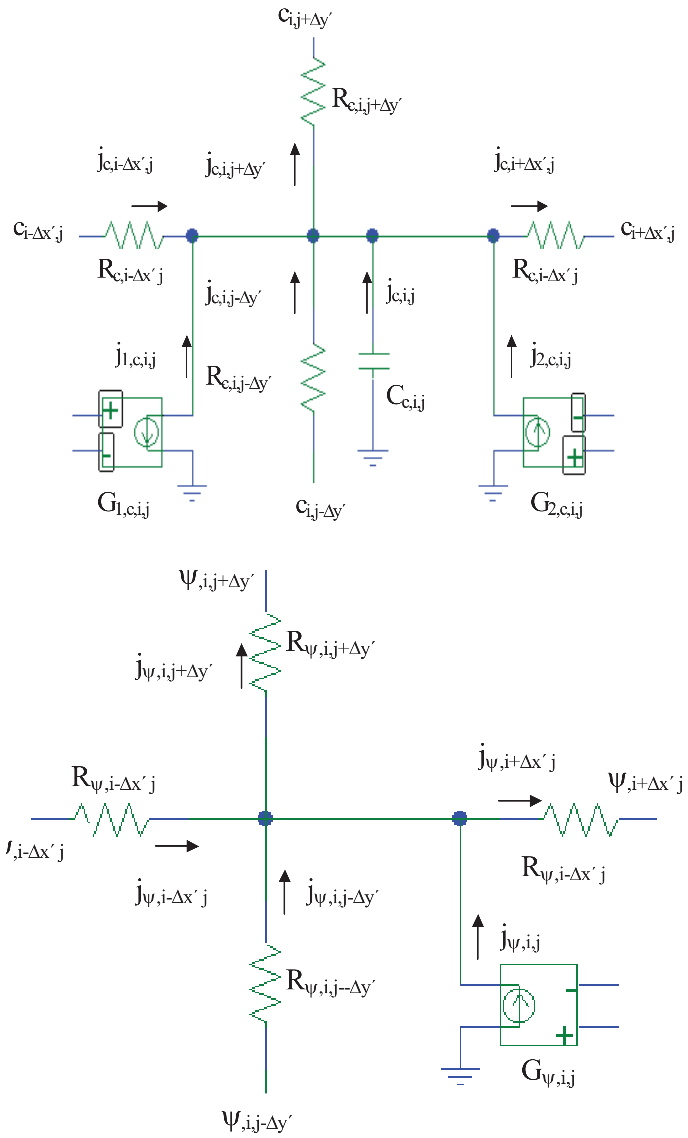

Figure 3. Network model of the volume element. a) stream function variable, b) salt concentration variable.

IV. Applications

A. Henry Problem

Two applications of the model were selected. A benchmark problem whose approximate analytical solution is well known and may be compared with that of our simulation to justify the accuracy of the propose model. The other application is and a real and complex aquifer. Henry and Biscayne aquifers are, respectively, the theoretical (benchmark) and real problems simulated by the network present model.

Henry's problem concerns a uniform, isotropic, rectangular aquifer, with initial salt concentration zero and impermeable top and bottom domain boundaries. It receives a constant freshwater recharge from the left side while the right side is exposed to a stationary body of seawater. Saltwater intrudes into the model domain through the lower right seaward, figure 1. As time progresses, saltwater enters the aquifer at the base of the right boundary (exit boundary) and mixes with freshwater (dispersion process), inducing a circulation from the floor of the sea into the zone of dispersion and back to the sea. After a long time, a final dynamic equilibrium state is reached.

The first authors to numerically reproduce the semi analytical solution proposed by Henry were Pinder and Cooper [18]. Later, using the stream function formulation, Lee and Cheng successfully compared their isochlor (c = 0.5) steady-state solution, with those reported by Henry and by Pinder and Cooper.

Due to the physically unrealistic Henry boundary condition, Segol et al. [19] introduced a new mixed boundary condition on the top portion of the seaward side to overcome the numerically difficulty caused by the original condition of Pinder and Cooper. More recently, other authors such as Croucher and O´Sullivan [1] Simpson and Clement [20] have used the new condition to verify their computational codes.

The numerical values of the parameters of the original Henry problem are:

Q = 6.6E-5 m^2/s, κ = 1.02E-9 m^2, μ = 1E-3 kg m^{-1} s^{-1}, g = 9.81 m/s^2, $\Delta\rho$ = 25 kg/m3, d = 1 m, l = 2 m, ε = 0.35, D = 6.6E-6 m^2/s, a = 0.263 and Pe = 10.

In the present study, the physical domain has been divided into 10 (vertical) × 80 (horizontal) volume elements. For this number of volume elements the total computing time is about 50 s in a Pentium 3 PC.

Figure 4 shows the isochlor concentration curves in the steady state, while the stream function curves are shown in Figure 5. These curves are provide by interpolation of the discrete results of Pspice, using Matlab. Table I compares the location of the isochlor, c = 0.5, with the recent (last) semi-analytical results of the original Henry problem by Simpson and Clement [21]. As can be seen, the accuracy of the present method is demonstrated.

TABLE I
Henry problem. Comparison of results

	Present work	Simpson and Clement 2004 [21]
y	x	x
0.0108	1.372	1.339
0.0995	1.402	1.419
0.1906	1.465	1.469
0.294	1.551	1.538
0.396	1.649	1.649
0.493	1.744	1.748
0.597	1.834	1.838
0.701	1.909	1.899
0.798	1.950	1.929
0.906	1.976	1.969
1	1.987	1.978

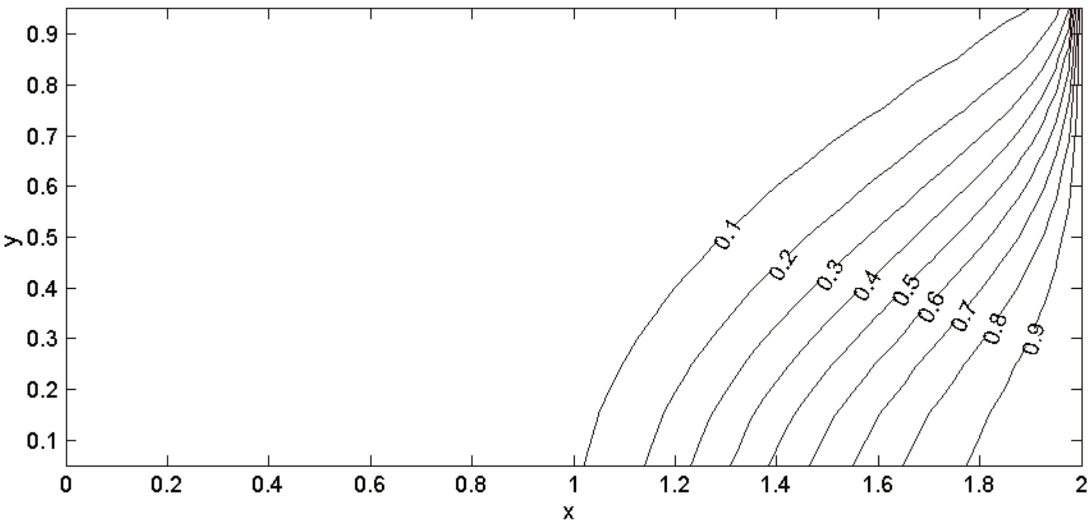

Figure 4. Henry problem. Steady state salt distribution. a 0.263, Pe 10.

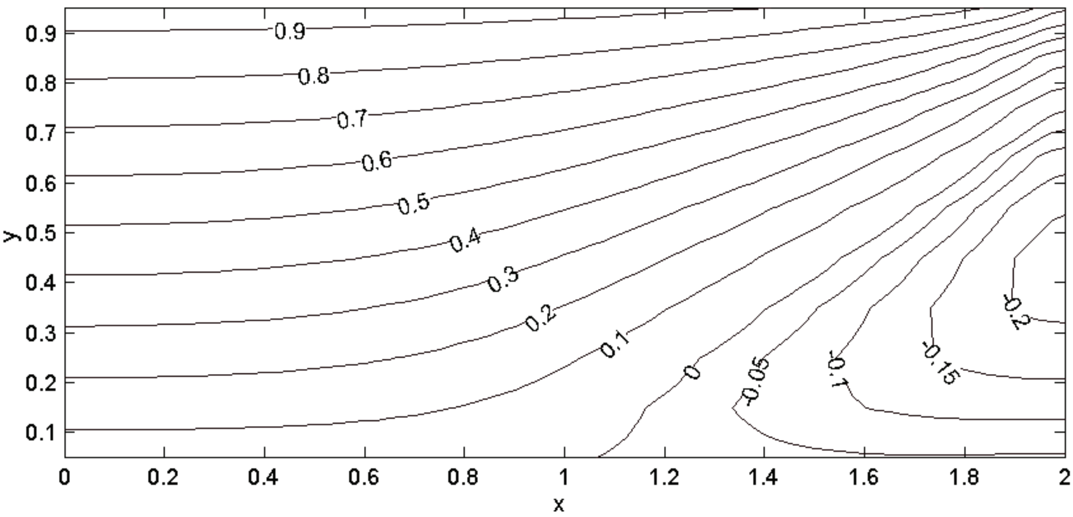

Figure 5. Henry Problem. Steady state stream function. a 0.263, Pe 10.

B Biscayne aquifer

Biscayne aquifer has been much studied since the early 1930s, particularly in relation to the mechanics and character of saltwater intrusion [5]-[11]. Located in southeastern Florida, its geological and hydrological aspects have been investigated and reported by Kohout [5]. Biscayne is a water table type aquifer extending from the land surface to an average depth of approximately 30 m below mean sea level.

The zone of diffusion is very thick and has a gradation of the salt content ranges from 16 ppm chloride (freshwater) to 19,000 ppm chloride (seawater). Part of the total seaward flow of the intruding seawater is discharged back to sea through the diffusion zone forming a salt water recirculation within the aquifer.

During the 1940s, saltwater intrusion was mitigated mostly through the use of temporary sheet-pile dam control structures. However, following a lengthy drought and prolonged period of uncontrolled drainage, the water supply in Miami-Dade County, experienced its greatest threat by saltwater contamination in 1945, when the lowest water levels were also reached. Salinity control structures were constructed by 1946 to prevent over drainage, increase water levels in the coastal part of the aquifer and allow the timely release of water during floods. Later, in 1970, new structures were constructed downstream and well withdrawals were reduced, both designed to diminish chloride concentrations measured by monitoring wells.

The schematics sketch of the Biscayne aquifer in the Cutler area is shown in Figure 6. Boundary conditions are also included in the figure. Impermeable condition is used on the base while hydrostatic condition is assumed far from the shore on the seaward side.

The numerical values of the geometry and parameters of this aquifer are:

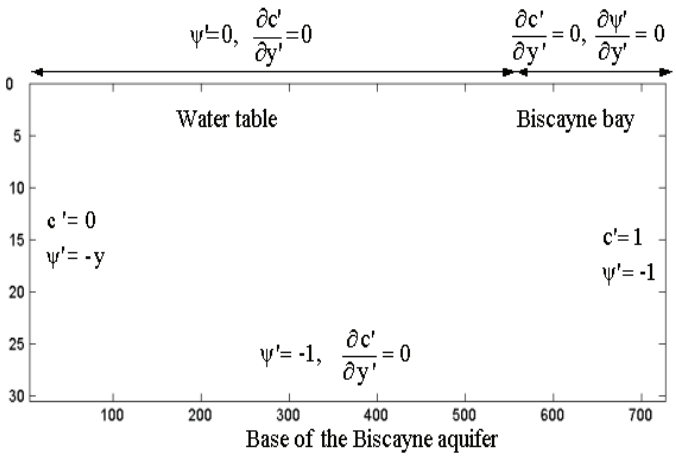

Figure 6. Sketch of the Biscayne aquifer and boundary conditions (vertical scale is exaggerated by a factor of 10).

l = 731.52 m, length of the Biscayne bay (see figure) = 182.88 m, d = 30.48 m, a = 0.04, Pe = 20÷200

Both the volume element of the physical domain and the total computing time have the same values (whatever the value of Pe) as in the case of Henry.

Figures 7 and 8, imported directly from the ambient of pspice, show the transient values of c and ψ for typical locations at the aquifer. Values of c were taken at the base, while values of ψ were taken for several elevations at the same x.

Steady state curves of iso-concentration are shown in Figure 9. For the range of small values of Pe, 20÷80 the effect of Peclet number is negligible but increases for Pe > 100. The convergence of the iteration process becomes increasingly more difficult as Pe increases, behaviour similar to that of the Reynolds number. As a result, numerical instabilities appear in the simulation.

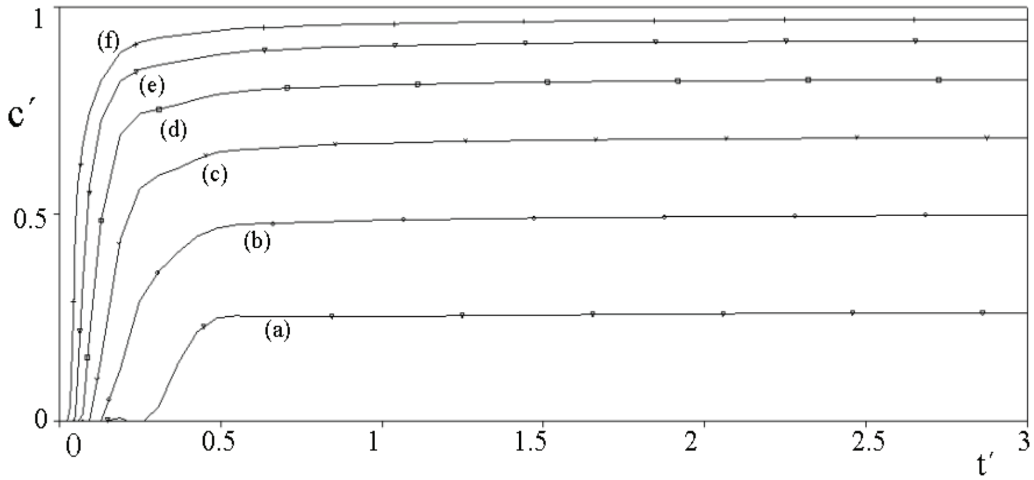

Figure 7. Transient values for salt concentration at the base of the aquifer. Pe 100
(a): x 416.0 m, (b): 434.3 m, (c): 452.6 m, (d): 470.9 m, (e): 489.2 m, (f): 507.4 m.

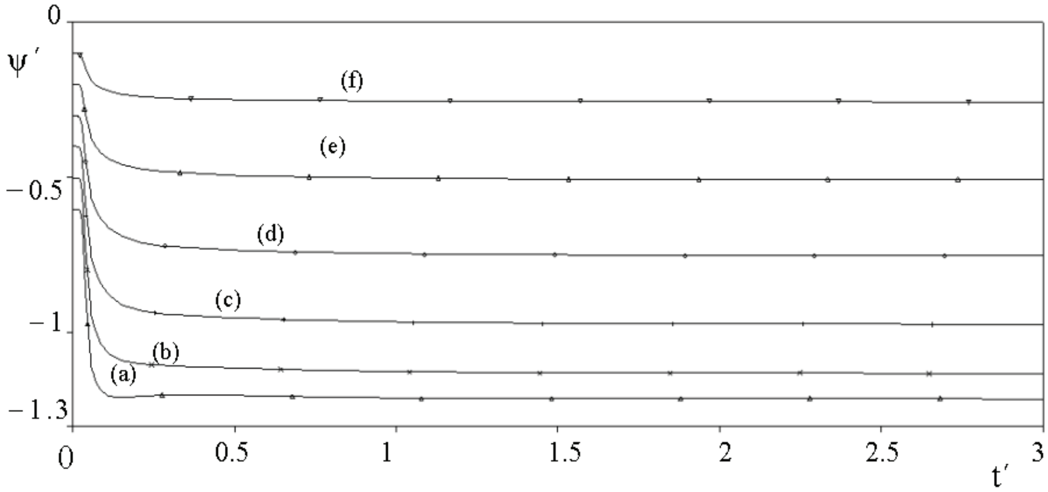

Figure 8. Transient values for stream function at x 500 m. Pe 100.
(a): y 3.048 m, (b): 6.096 m, (c): 9.144 m, (d): 12.19 m, (e): 15.24 m, (f): 18.29 m.

As regards stream function, figure 10 shows the steady state of ψ for the same values of Pe. The effects mentioned above for salt concentration also apply to the variable ψ.

A significant result in this type of problem is the location of the toe of salt water recirculation for the isochlor c = 0.5 at the bottom of the aquifer. Table II shows this information for as a function of Pe.

Table III shows the comparison between our results of the location of salt water recirculation for different values of c, with those of Lee and Cheng [6]. Appreciable deviations of these values appear due to some kind of numerical dispersion according to Frind [22].

The ψ = -1 streamline separates the region of saltwater recirculation (ψ<-1), which contains streamlines that begin and end the Biscayne bay, from the upper region of no-recirculation (ψ>-1), which contains fresh water through-flow, and meets the aquifer base at the stagnation point x = 406.7 (376.5 for Lee and Cheng).

TABLE II
LOCATION OF THE TOES OF SALTWATER RECIRCULATION. C 0.5

Seaward distance	248	270	286	292	297	317
Pe	20	40	60	80	100	200

TABLE III
COMPARISON OF THE TOES OF SALTWATER RECIRCULATION. PE_E 100

c	0	0.25	0.5	0.75	0.99	1
Present method	399.6	414.5	434.9	460.1	549.0	731.5
Lee and Cheng [6]	324.2	373.6	405.1	435.6	-	499.5

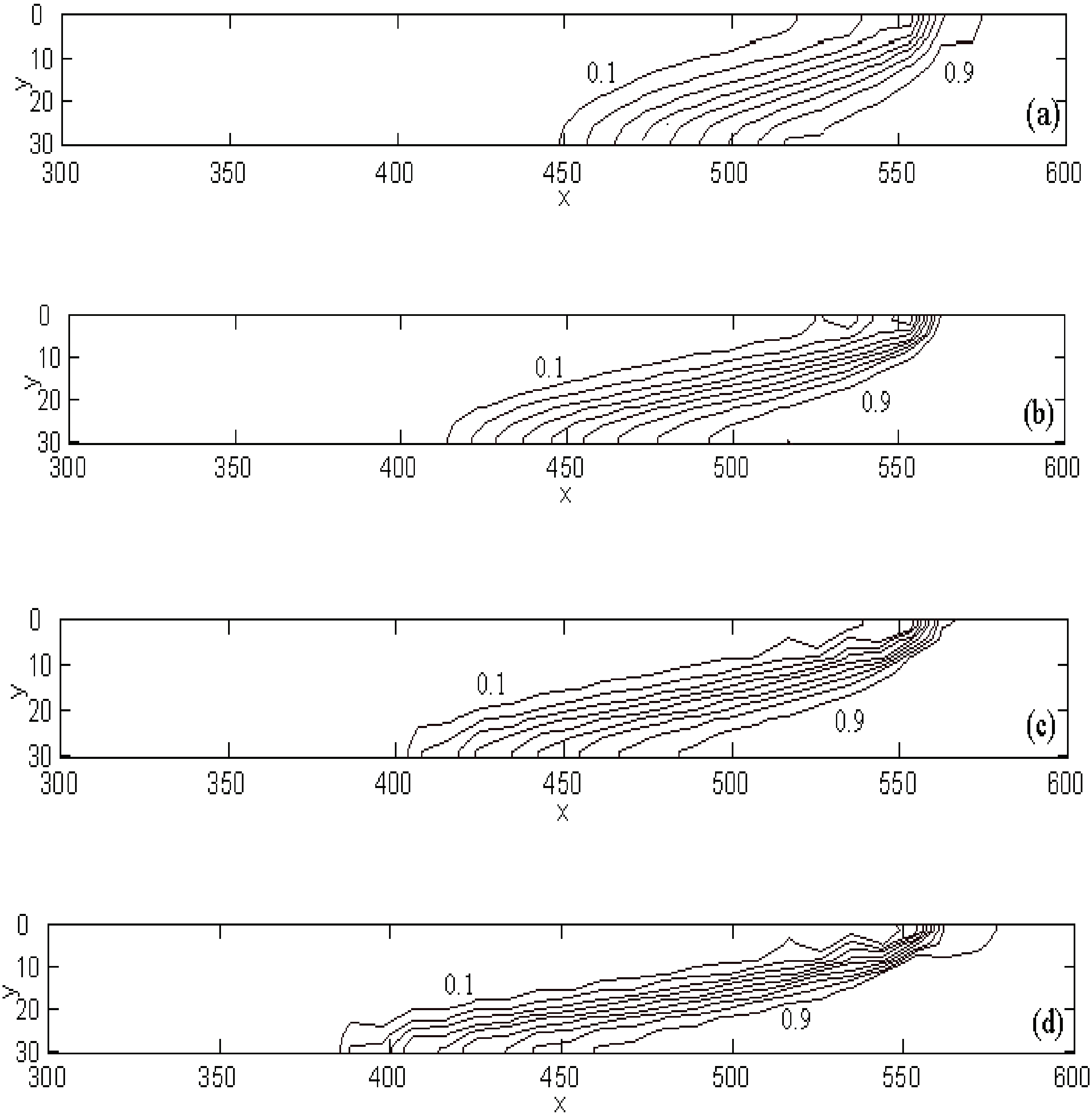

Figure 9. Biscayne problem. Steady state salt distribution, a 0.04. Concentration step between adjacent curves: 0.1. (a): Pe 20, (b): Pe 60, (c): Pe 100, (d): Pe 200.

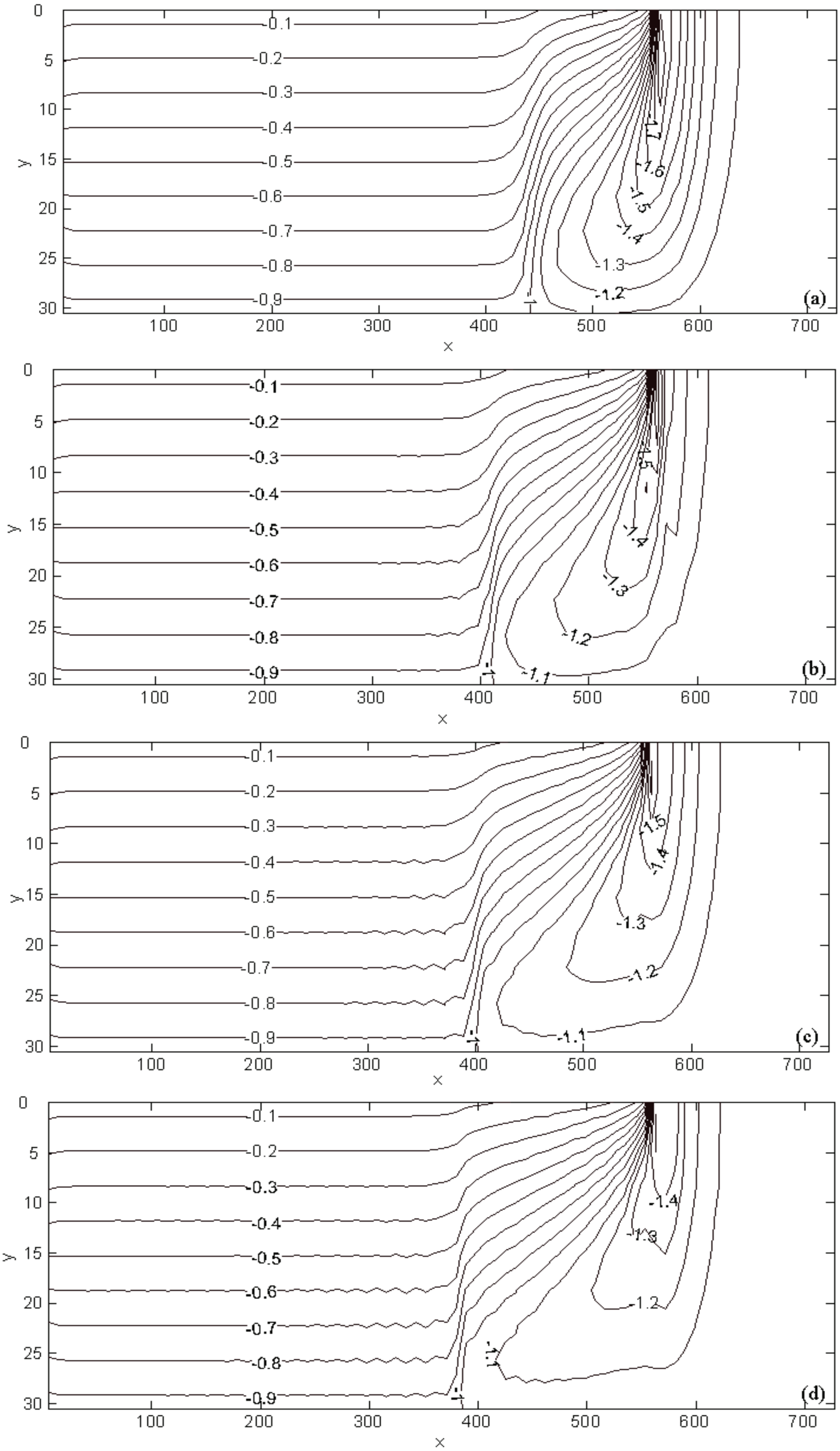

Figure 10. Biscayne problem. Steady state stream function. a 0.04. (a): Pe 20, (b): Pe 60, (c): Pe 100, (d): Pe 200.

V. Conclusion

For the first time, stream function formulation is applied to simulate the dynamic of a real aquifer (Biscayne, Florida). The proposed numerical model is based on the network simulation method - already used to simulate other nonlinear problems of similar complexity in science and engineering. In order to demonstrate the accuracy and efficiency of the proposed model, transient streamlines and isochlor contours, for a relatively small number of volume elements, are compared with those provided by other classical numerical methods based on other formulations. The influence of the Peclet number, which reaches critical values in this case in the final distribution of streamlines is studied, as well as the locations of the toes of saltwater recirculation.

References

[1] A. E. Croucher and M. J. O'Sullivan, "The Henry problem for saltwater intrusion", in *Water Resour. Res*., vol. 31(7), 1995, pp. 1809 1814.

[2] E. Holzbecher, *Modelling Density-driven Flow in Porous Media*. Berlín. Springer-Verlag, 1998, ch. 11.

[3] H. Gotovac, R. Andricevic, B. Gotovac, V. Kozulic and M. Vranjes, "An improving collocation method for solving the Henry problem", in *Journal of Contaminant Hydrology*, vol. 64, 2003, pp. 129-149.

[4] C.F González-Fernández and F. Alhama, *Heat Transfer and the Network Simulation Method*, Trivamdrum-India. Research Signpost J. Horno. Ed. 2002, ch. 2.

[5] F. A. Kohout, "Flow pattern of freshwater and salt water in the Biscayne aquifer of the Miami area, Florida", in *Int. Ass. Sci. Hydrol. Publ*., vol. 52, 1960, pp. 440-448.

[6] C. H. Lee and R. T. Cheng, "On seawater encroachment in coastal aquifers", in *Water Resour. Res*., vol. 10(5), 1974, pp. 1039-1043.

[7] S. D. Leach, Klein, Howard and E. R. Hampton, Hydrologic effects of water control and management of southeastern Florida: *Florida Geological Survey*. *Report of Investigations*, 60, 1972, p. 115.

[8] J. L. Hughes, Saltwater-barrier line in Florida: Concepts, considerations, and site examples. *U.S. Geological Survey Water-Resources Investigations Report* 79-75, 1979, p. 29.

[9] M. L. Merritt, Numerical simulation of a plume of brackish water in the Biscayne aquifer originating from a flowing artesian well, Dade County, Florida. *U.S. Geological Survey Water- Supply Paper* 24-64, 1996, p. 74.

[10] R. A. Renken, J Dixon, J. Koehmstedt, S. Ishman, A. C. Lietz, R. L. Marella, P. Telis, J. Rogers and S. Memberg. Impact of anthropogenic development on coastal ground-water hydrology in southeastern Florida, 1900-2000. *U. S. Geological Survey*, Circular 1275, 2002, 138 p.

[11] Langevin, C.D., Simulation of ground-water discharge to Biscayne Bay, southeastern Florida: *U.S. Geological Survey Water-Resources Investigations Report*, 00-4251, 2001, 127 p.

[12] J. Horno, C. F. González-Fernández and A. Hayas, "The network method for solutions of oscillating reaction-diffusion systems", in *Comp. Phys. Comm*., vol. 118, 1995, pp. 310-319.

[13] F. Alhama, J. Zueco and C. F. González-Fernández, "An inverse determination of unsteady heat fluxes using a Network Simulation Method", in *J. Heat Transfer*, vol. 125, 2003, pp.1178-1183.

[14] F. Alhama and C. F. González-Fernández, "Transient thermal behaviour of phase change processes in solid foods with variable thermal properties", in *J. Food Eng*., vol. 54(4), 2002, pp. 331-336.

[15] J. F. López Sánchez, F. Alhama and C. F. González-Fernández, "Introduction and performance of species in a diffusive Lotka-Volterra system with time dependent coefficients", in *Ecological Modelling*, vol 183, 2005, pp.1-9.

[16] A. Soto Meca, S. A. Gómez-Lopera, F. Alhama and C. F. González-Fernández, "A new simple model to numerically simulate refreshing processes in saline aquifers," in *Proc. 18th SWIM*, Cartagena, 2004, p. 54.

[17] PSPICE 6.0, "Microsim Corporation Fairbanks", Irvine, California 92718, 1994.

[18] G. F. Pinder and H. H. Cooper Jr., "A numerical technique for calculating the transient position of the saltwater front", in *Water Resour. Res*., vol. 6(3), 1970, pp. 875 882.

[19] G. Segol, G. F. Pinder and W. G. Gray, "A Galerkin-finite element technique for calculating the transient position of the saltwater front", in *Water Resour. Res*.,vol. 11(2), 1975, pp 343 347.

[20] M. J. Simpson and T. P. Clement, "Theoretical analysis of the worthiness of the Henry and Elder problems as benchmarks of density-dependent groundwater flow models", in *Adv. Water Resour*., vol. 26, 2003, pp.17 31.

[21] M. J. Simpson and T. P. Clement, "Improving the worthiness of the Henry problem as a benchmark for density-dependent groundwater flow models", in *Water Resour. Res.*, vol. 40, 2004, pp.1-11.

[22] E. O. Frind, "Simulation of long-term transient density-dependent transport in groundwater", in *Adv. Water Resour*., vol 5, 1982, pp. 73-88.